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IDENTIFICATION AND CONTROL OF A FLEXIBLE LAUNCH VEHICLE

by Rob Roy and K. W. Jenkins

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RENSSELAER POLYTECHNIC INSTITUTE
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ABSTRACT

The problem of controlling a system described by a set of difference equations when some of the system parameters vary from their nominal value is analyzed. The system has both statistical and deterministic disturbances acting upon it. A control is to be synthesized which regulates the state of the system in an optimal fashion, as defined by a quadratic performance index, using only the available noisy measurements.

The control system is arbitrarily chosen to comprise a control loop and a parameter identifier. The identifier estimates the unknown plant parameters from normal observations of the plant input and output. The control loop alters its policy in accordance with these new parameter values. The control loop is obtained from a Dynamic Programming derivation which accounts for the deterministic as well as the statistical disturbances. The resulting system is comprised of a least squares state estimator, a feedback gain matrix operating on this estimate and a feedforward input term.

The general type of parameter identifier considered was of the least squares type. Two methods of using

least squares techniques are extensively analyzed. The chosen system uses a differential corrections method generically similar to a Kalman filter. Because the identifier works with the system state equations augmented by the parameters, state estimates are also generated. These are used in the control loop. The new parameter information is used to recompute the gain matrix one step backward from the nominal Riccati matrix. For a linear system the identification method can be shown to converge whenever the percentage errors in the parameters are sufficiently small.

The method of identification and control was applied to the pitch control of a large flexible launch vehicle. Body mounted pitch and pitch rate gyros are the only sensors. The vehicle model incorporated third order rigid body equations plus first bending mode. The control is to maintain pitch attitude and minimize the bending vibration in the face of steady wind shears and gusts. The entire system was evaluated on a digital computer. Without the parameter identification a 20% reduction from nominal in the bending frequency caused severe vibration. With parameter identification, this vibration was much reduced. Steady pitch angle was maintained to less than one degree for worst case wind conditions over the flight interval.

SECTION I

1.1 Introduction

The words "adaptive control" seem to have a high emotional content for control systems engineers. They imply a system which, like a man, can adjust itself to a changing environment. The adaptive system will therefore be the system designer's magnum opus. Once completed, he need no longer be concerned with what the controlled process does; the adaptive system will figure it out and take steps. Exactly how such a system is to be designed in the first place seems to take a little longer, and there is a faint suspicion that the unmodified concept of adaption is a chimera. The work reported here is loosely referred to as "adaptive", but in a rather specialized sense. The analysis which is developed arose from a problem suggested to the author and Dr. Rob Roy by M. Borelli of NASA. It is the one analyzed in Section IV. However, as is often the case, consideration of a specific problem led to a more general method than the original problem required. Roughly, the control problem had the following characteristics:

1. Only certain measurements of the process were allowed by the physics of the situation and

the available engineering technology.

2. Although a good mathematical approximation of the process behavior was available, certain parameters of that description could not be accurately forecast due to engineering and economic limitations.

3. The control problem was not "easy" in the sense that the process description was complex and the performance requirements were stringent.

4. State and measurement disturbances were present which had to be accounted for.

It is clear that such a set of features is common to a wide class of systems.

Although conventional feedback is known to possess the ability to handle problems of the sort listed, the general control structure shown in Figure 1.1 was postulated. It is comprised of a feedback controller which processes the measurable outputs and generates a control input based upon them. A parameter identifier also observes the outputs of the plant as well as the control inputs. Based upon these observations, new parameter estimates are generated and fed

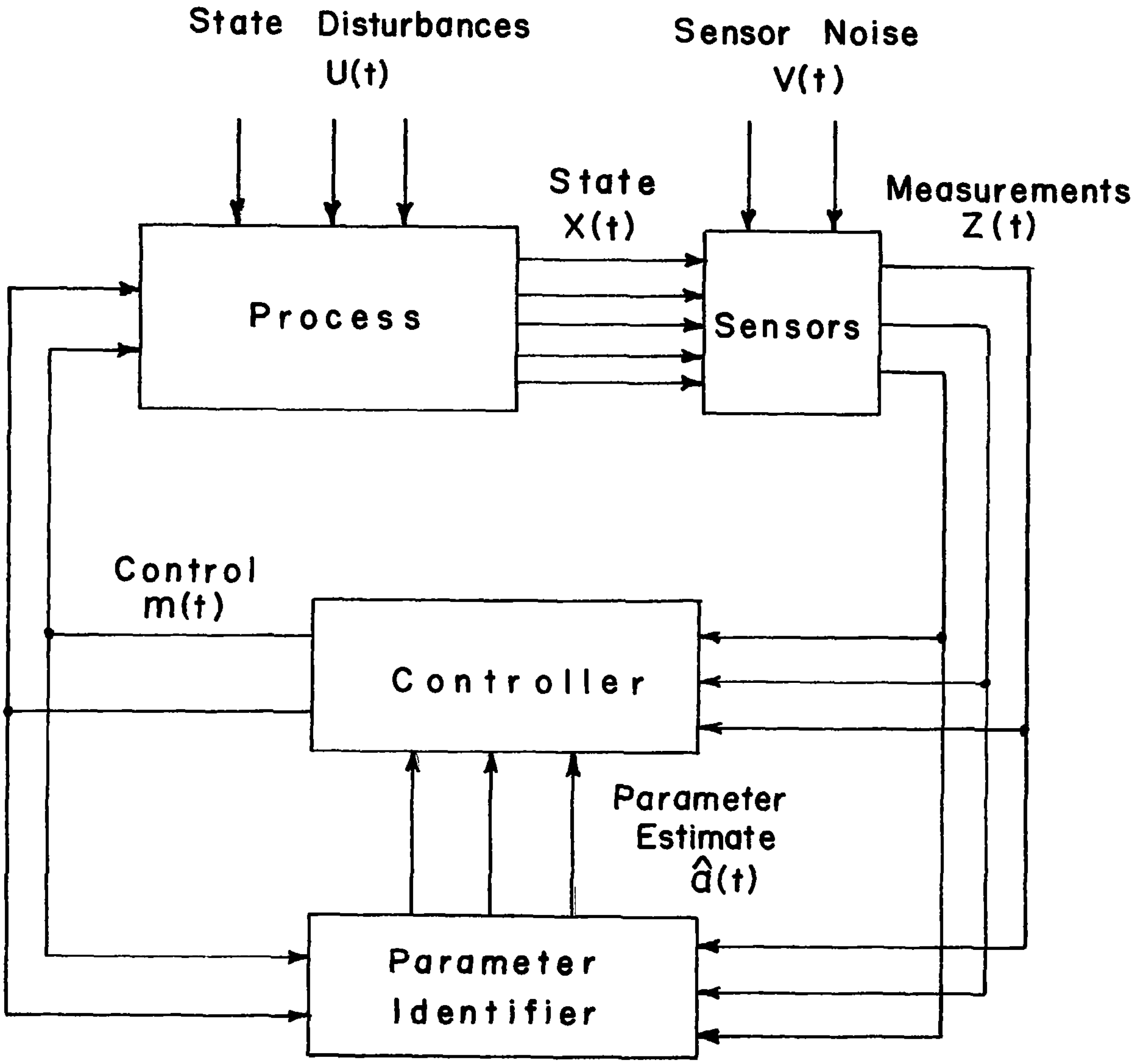

Figure 1.1 Adaptive System Structure

to the controller which in some fashion alters its control law in accordance with these estimates. It is this feature of the proposed system which leads the author to use the word "adaptive". Whether this is a "best" structure for an adaptive controller is certainly open to debate. Horowitz[14*], for instance, feels otherwise.

For reasons of personal preference and esthetics, the identifier was to function with limited storage capacity and finite computing speed. Since it is assumed from the structure of the controller that the identification is performed on line, those identifiers using the entire measurement history or correlation techniques are excluded. The control law proper is to be designed based upon the developments of optimal control theory. This decision was based upon the great flexibility and generality of these methods together with the fact that a feedback controller usually results from their use. Because a feedback control is directly derivable from optimal control theory for the case where the process equations are linear, attention is restricted to that case. Further, because digital controllers

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[*] Throughout this thesis, superscript numbers refer to the similarly numbered items in SECTION VI, LITERATURE CITED.

utilize difference equations and most estimation literature is written in terms of discrete measurements, the process is assumed to be described by a vector difference equation. The corresponding results for the continuous measurement case are indicated in the second Appendix.

1.2 Historical Review

The development of optimal control theory is partially a recognition and use of the Calculus of Variations to solve control problems. Bellman[3] provided an alternate viewpoint for the problem of minimizing a performance index subject to a differential constraint similar to the Markov property of statistics. Pontriagin[23] and his co-workers developed powerful extensions to the Calculus of Variations and Merriam[22] has been in the forefront in applying computational methods to the solution of optimal control problems. The primary advantage of optimal control lies in the formalized design procedure. A measure of performance is set up and the control system which is "best" in the sense of maximizing the performance measure results directly from the mathematical manipulations.

Although the Calculus of Variations was applied to deterministic systems, interest soon arose in optimization of systems driven by random disturbances and subject to measurement noise. Bellman referred to such problems as the "Stochastic Control Problem", and considerable work has been done on this class of problems. Florentin[11] expanded "Dynamic Programming" to include the stochastic control problem, and derived the stochastic Hamilton-Jacobi equation. For linear systems with quadratic error measure, he was able to derive the feedback gain matrix of the controller. Kalman[16, 17] brought to the attention of the control engineer the possibilities of using the Least Squares estimation techniques from statistics for the estimation of the system state given noisy measurements. Joseph and Tou[15] paralleled Kalman's use of orthogonal projections to show that the solution of the stochastic control problem for linear systems with Gaussian noise and quadratic error measure was a Kalman estimator of the state driving the conventional feedback gain matrix of optimal control theory. Although these developments assumed uncorrelated or "white" noise, results for colored noise have been obtained by Bryson and Johansen[6]. The entire least squares estimation problem with correlated measurements has also been independently derived by Battin[2] in connection with the problem of differential corrections to trajectories. All of the control work, however, considers only the regulator problem and assumes zero mean disturbances.

The use of estimation techniques to fit models of processes to observed data has attracted considerable current interest from control engineers. Use of the Least Squares method dates from Levin[21] who estimated the impulse response of a system with noisy measurements. Kerr and Surber[29] considered observation of the system during normal operation and showed how to compute the expected mean square error of the identification. Surber[26] later applied gradient techniques to the problem of fitting a model to data. Kopp and Orford[18] used linear regression analysis to identify the parameters of a second order system. The control law was modified in an algebraic relation to the parameter change to maintain the transient response invariant relative to a model second order system. Lee[19] used Least Squares for systems with input noise only. He also gives a good summary of the relations of filtering and identification. Recently, Cox[30] has used Dynamic Programming to estimate state variables in non-linear cases. The resulting equations are not easily solved, however.

The problem to which the method is applied is the pitch control of a large flexible launch vehicle. Due to the vehicle flexure, the pitch and pitch rate sensors measure both rigid body motion and local flexure. Further, the bending modes cannot be overly excited by the system or the vehicle will break up. This problem is of great current interest as might be imagined. Many approximate schemes of taking out the flexure effects from the measurements have been proposed. Tutt and Wyameyer[27] use a model of the vehicle; Smyth and Davis[25] propose notch filters with adjustable center frequency to take out the bending. Lee[20] uses a redundant gyro to try and cancel out the bending in the measurements. Of these, only the notch filter approach has had much engineering success and it depends upon bending frequencies being higher than the speed of response of the closed loop system. For this study only first order bending and no slosh modes are included. Pitch and pitch rate gyros are assumed to be the only sensors. State noise enters as wind gusts along with a steady wind which includes an angle of attack disturbance.

SECTION II

CONTROL OF LINEAR PLANTS SUBJECT TO DISTURBANCES

This section studies the problem of regulating a linear plant which is subjected to state disturbances of both a random and deterministic nature. Moreover, it is assumed that the state cannot be measured exactly, and that such measurements as are available are corrupted by random noise.

For strictly random disturbances of a certain form the result has been known for some time. Joseph and Tou[15] showed that for white noise of mean zero the optimal controller is comprised of an optimal state estimator in the form of a Kalman Filter[16], and the usual feedback gain matrix which results in the noise free case where all states are measurable. However, their derivation followed Kalman in using the projection theorem which lends little insight into how the problem at hand can be generalized.

As might be expected, the presence of deterministic state disturbances does not change the feedback gains nor the filter constants but causes a feedforward term to appear which acts to balance out the known disturbance. The deterministic measurement disturbance simply appears as an additive term in the filter plant model.

2.1 Control with Stochastic Disturbances

The problem of minimizing a quadratic performance index subject to both stochastic and deterministic disturbances and measurement errors will be analyzed mathematically in this Section. The plant is described by the linear difference equation

$$x(k+1) = A(k)\ x(k) + B(k)\ m(k) + \Gamma(k)\ u(k) + d_1(k) \tag{2.1-1}$$

The state x cannot be measured directly, rather the noisy measurement vector z related to the state by the linear relation

$$z(k) = C(k)\ x(k) + L(k)\ v(k) + d_2(k) \tag{2.1-2}$$

is available for control purposes. The control input m(k) is to be manipulated so as to achieve a minimum of the cost function

$$J = \frac{1}{2} \sum_{k=0}^{N-1} \{ x^T(k+1) \quad (k+1)\ x(k+1) + m^T(k)\ Q(k)\ m(k) \} \tag{2.1-3}$$

The disturbances in equations 1 and 2 have been segregated into the deterministic ones, $d_1(k)$ and $d_2(k)$, and the random ones, u(k) and v(k). Since any random variable with non-zero mean is the sum of the mean plus another random variable of zero mean it is not restrictive to assume that u(k) and v(k) have a mean of zero. The second order statistics of u(k) and v(k) are assumed as unit covariance matrices. Compactly this is written

$$E\{u(k)\, u^T(j)\} = \delta_{kj} \qquad E\{v(k)\, v^T(j)\} = \delta_{kj}$$

$$E\{u(k)\, v^T(j)\} = 0 \tag{2.1-4}$$

So that u and v are assumed to be independent white sequences. Since the noise gain matrices L and Γ can account for any other variance, the unit variance assumption is also not restrictive. On the other hand, the use of white sequences is a definite limitation. One method of handling correlated noise is to adjoin the necessary filters which produce that noise from a white sequence to the plant description. This however produces a system which is not completely controllable. Other authors[2, 6] have considered the estimation problem and obtained results directly. This analysis will be restricted to the statistics given by (2.1-4). Figure 2-1 shows a block diagram of the system to be controlled.

In order to maintain a meaningful estimation

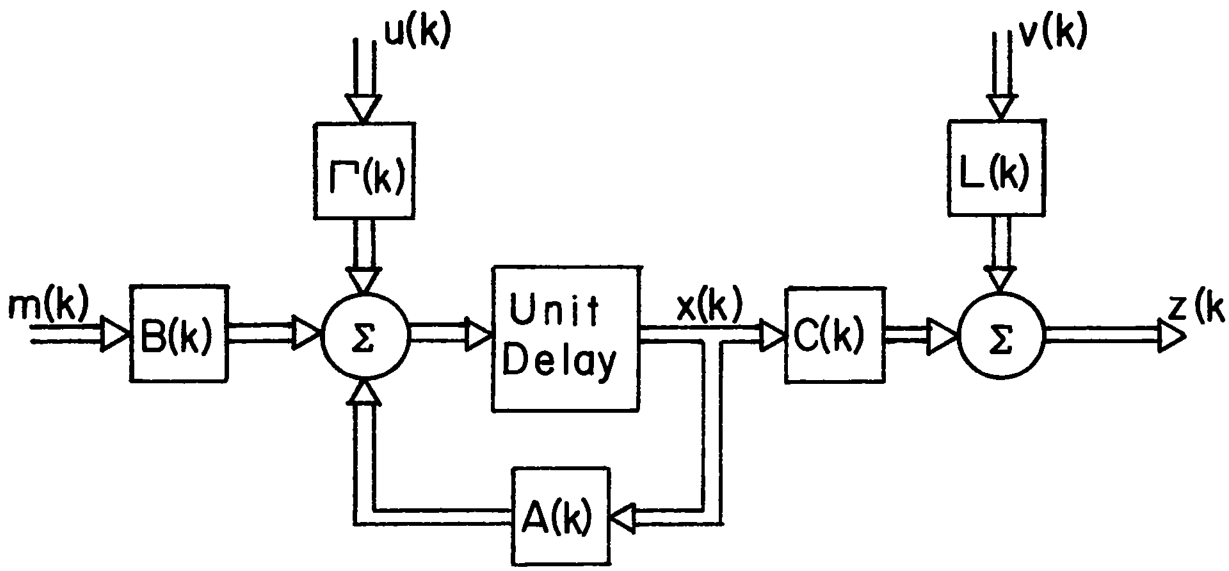

Figure 2.1 Linear Discrete Plant

problem even when the state noise is identically zero, the initial conditions of the plant are assumed drawn from a random population such that

$$E\{x(0)\} = \bar{x}(0)$$

$$\operatorname{cov}\{x(0)\} = P_0 \qquad (2.1\text{-}5)$$

Because the criterion function J contains the states x(k), it is itself a random variable. In this case it is therefore more meaningful to work with the mean of J averaged over the joint distribution of u, v and x(0). However, by Bayes Theorem, that is equivalent to averaging first over the disturbances, given the initial conditions and then averaging over the initial conditions. This is done in order to obtain the plant structure.

To apply Dynamic Programming it is necessary to begin by defining the function

$$J_k(x(k)) = \underset{m}{\text{Min}} \; \underset{u,v|x(0)}{E} \left[\frac{1}{2} \sum_{i=k}^{N-1} x^T(i+1)\, S(i+1)\, x(i+1) + m^T(i)\, Q\, m(i) \right] \qquad (2.1\text{-}6)$$

From the principle of optimality, the minimum cost over any future control interval is a function only of the present state and the future control. This statement holds for the stochastic cases since the cost function is an average quantity. Mathematically the principle is written

$$J_k(x(k)) = \underset{m}{\text{Min}} \underset{u,v|x(0)}{E} \left\{ \frac{1}{2} x^T(k+1)\, S\, x(k+1) + \frac{1}{2} m^T(k)\, Q\, m(k) + J_{k+1}(x(k+1)) \right\} \tag{2.1-7}$$

The boundary condition for the recursive relation given by (2.1-7) is

$$J_N(x(N)) = 0 \tag{2.1-8}$$

In order to obtain a solution of (2.1-7) the quantity $J_{k+1}(x(k+1))$ will be assumed to have the form

$$2\, J_{k+1}(x(k+1)) = \left[A(k+1)\, x(k+1) + \beta(k+1)\right]^T G \left[A(k+1)\, x(k+1) + \beta(k+1)\right] + \gamma_{k+1} \tag{2.1-9}$$

The matrix G is assumed to be positive semi-definite and γ_{k+1} is a non-negative scalar. The vector $\beta(k+1)$ has no restrictions as to its components. Equation (2.1-9) may also be written as

$$2\, J_{k+1}(x(k+1)) = x^T(k+1)\, F(k+1)\, x(k+1) + x^T(k+1)\, \zeta(k+1) + \zeta^T(k+1)\, x(k+1) + \xi(k+1) \tag{2.1-10}$$

by defining the quantities

$$F(k+1) = A^T(k+1)\ G\ A(k+1)$$

$$\zeta(k+1) = A^T(k+1)\ G\ \beta(k+1) \qquad (2.1\text{-}11)$$

$$\xi\ (k+1) = \beta^T(k+1)\ G\ \beta(k+1)$$

The assumed form of J_{k+1} given by (2.1-10) may now be substituted back into equation (2.1-7) to obtain the cost function in terms of x(k + 1) and m(k). Substitution of the plant equation (2.1-1) for x(k + 1) gives $J_K(x(k))$ as

$$J_k(x(k)) = \underset{m(k)}{\text{Min}} \sum_{u,v\,|\,x(0)} \frac{1}{2}\Big[\big[A\ x(k) + B\ m(k) + \Gamma u(k) + d_1(k)\big]^T R(k+1)\ \big[A\ x(k) + B\ m(k) + \Gamma\ u(k) + d_1(k)\big] + \zeta^T(k+1)\big[A\ x(k) + B\ m(k) + d_1(k)\big] + \big[A\ x(k) + B\ m(k) + d_1(k)\big]^T \zeta(k+1) + m^T(k)\ Q\ m(k) + \xi(k+1)\Big] \qquad (2.1\text{-}12)$$

In obtaining this result, use has been made of the fact that the mean of u is zero. The matrix R(k + 1) has been defined for convenience as

$$R(k+1) = S(k+1) + F(k+1) \qquad (2.1\text{-}13)$$

In order to separate the control and the estimation problem the terms involving the control m(k) must be segregated. Equation (2.1-12) can be written, by completing the square in terms of m(k) and again making use of the zero mean of u(k), in the form

$$J_k(x(k)) = \underset{m(k)}{\text{Min}} \; \underset{u,v|k\;(0)}{E} \; \frac{1}{2} \Bigg[\left[Q + B^T R B\right] m(k)$$

$$+ B^T R \left[A\, x(k) + d_1 + R^{-1} \zeta\right]^T \left[Q + B^T R B\right]^{-1}$$

$$+ \left[A\, x(k) + d_1 + R^{-1} \zeta\right]^T R - R B \left[Q + B^T R B\right]^{-1} B^T R$$

$$\left[A\, x + d + R^{-1} \zeta\right] - \zeta^T R^{-1} \zeta + \xi(k+1) + u^T \Gamma^T R\, \Gamma\, u \Bigg] \tag{2.1-14}$$

Equation (2.1-14) places in evidence the quadratic dependence of J_k upon the first term. Since it is the only one containing the control m(k) the choice of m(k) must be such as to minimize the magnitude of that term. The correctness of the form of (2.1-14) depends upon the existence of an inverse for the matrices $\left[Q + B^T R B\right]$ and R. Actually it is not necessary that R be positive definite since the R^{-1} term is only used for convenience. This term arises when the $\zeta(k+1)$

term is included with the Ax + d quantity to complete the square. If R is symmetric then it may always be written as the product of two matrices $R = M^T M$ [7] and by expanding in terms of $MAx + Md_1 + \zeta$ the same result is obtained. The other inverse is more serious. Its presence requires that either Q or R be positive definite. Assume for the moment that this is so.

In order to separate the control and the estimation problem, write the control as

$$m(k) = - K(k)\, \hat{x}(k) - h(k) \tag{2.1-15}$$

where

$$K(k) = \left[Q(k) + B^T R(k+1) B\right]^{-1} B^T R(k+1) A(k)$$

$$h(k) = \left[Q(k) + B^T R(k+1) B\right]^{-1}$$

$$B^T \left[R(k+1)\, d_1(k) + \zeta(k+1)\right] \tag{2.1-16}$$

Define the error term

$$\tilde{x}(k) = x(k) - \hat{x}(k) \tag{2.1-17}$$

Then the performance index $J_k(x(k))$ is minimized with respect to m(k) if

$$\underset{u,v\,|\,x(0)}{E} \frac{1}{2} \quad \tilde{x}^T(k)\, K^T(k) \left[Q + B^T R B\right] K(k)\, \tilde{x}(k) \tag{2.1-18}$$

is minimized with respect to $\tilde{x}$. But this is precisely the form of the estimation problem as formulated by Kalman and others. Choose an estimate of the state $\tilde{x}(k)$ which minimizes the weighted sum of the square errors.

Define the matrix

$$H(k) = K^T(k) \left[Q(k) + B^T R(k+1) B \right] K(k) \qquad (2.1\text{-}19)$$

and denote the covariance of the estimate error by

$$E \{ \tilde{x}(k) \tilde{x}^T(k) \} = P(k) \qquad (2.1\text{-}20)$$

If R is symmetric so is H and

$$\underset{u \mid x(0)}{E} \{ \tilde{x}^T H \tilde{x} \} = tr \{ P H \} \qquad (2.1\text{-}21)$$

Then, the optimal cost function is written

$$J_K(x(k)) = \left[A x(k) + d_1(k) + R^{-1}(k+1) \zeta(k+1) \right]^T$$
$$\left[R(k+1) - R(k+1) B \left[Q + B^T R B \right]^{-1} B^T R(k+1) \right]$$
$$\left[A x(k) + d_1(k) + R^{-1}(k+1) \zeta(k+1) \right] + \xi(k+1)$$
$$- \zeta^T(k+1) R^{-1} \zeta(k+1) + tr \{ \Gamma \Gamma^T R(k+1) \} + tr \{ P H \}$$
$$(2.1\text{-}22)$$

Comparison of (2.1-22) to the assumed form for J reveals the following recursive relationships

$$R(k+1) = S(k+1) + F(k+1)$$

$$K(k) = \left[Q(k) + B^T R(k+1) B\right]^{-1} B^T R(k+1) A(k)$$

$$F(k) = A^T(k) R(k+1) \left[A(k) - B K(k)\right] \qquad (2.1\text{-}23)$$

and

$$\alpha(k) = R(k+1) d_1(k) + \zeta(k+1)$$

$$h(k) = \left[Q(k) + B^T R(k+1) B\right]^{-1} B^T \alpha(k)$$

$$\zeta(k) = A^T(k) \left[\alpha(k) - R(k+1) B h(k)\right] \qquad (2.1\text{-}24)$$

These relations completely specify the quantities K(k) and h(k) in the control law given by (2.1-15). The boundary conditions are obtained by applying the terminal condition of equation (2.1-8).

$$F(N) = 0$$

$$\zeta(N) = 0 \qquad (2.1\text{-}25)$$

With the recursive relations in hand it is easily verified that F is symmetric and positive definite if Q is positive definite or S is positive definite and B is of rank r, where r is the number of control inputs. Under these conditions the required matrix inversions may be carried out. This solves the control portion of the problem. But this solution requires an estimate $\hat{x}(k)$ of the state which minimizes the quadratic form of equation (2.1-18).

2.2 State Estimation

The results of Section 2.1 depend upon choosing an estimate $\hat{x}$ which minimizes the quadratic form of equation (2.1-18). This problem has been solved by many workers. In the control literature Kalman[16, 17] is frequently referred to because of familiarity, however the result is actually from probability theory. In probability theory, the applicable lemma[8] is

Lemma: Let $\rho(x)$ be a symmetric positive definite convex function such that $\rho(0) = 0$ and let $x(k)$ be a random vector whose probability distribution is symmetric about the mean. The estimate $\hat{x}(k)$ of $x(k)$ based upon the measurements $y(0)$ to $y(k)$ which minimizes $E\{\rho(\hat{x} - x)\}$ is the conditional expectation

$$\hat{x}(k) = E\{x(k) \mid y(0) \ldots y(k)\} \tag{2.2-1}$$

Notice that the form of equation (2.1-18) meets the requirements of the lemma. Suppose that the state and measurement noises are Gaussian. Then the conditional distribution is also Gaussian and the problem is identical to that considered by Kalman except for the deterministic measurement error which must be subtracted out.

The results are in the form of the recursive relations.

$$x(k+1 \mid k+1) = x(k+1 \mid k) + \psi(k+1) \Big[z(k+1) - C\,x(k+1 \mid k) - d_2(k) \Big]$$

$$\psi(k+1) = P(k+1 \mid k)\; C \left[L\,L^T + C^T\,P(k+1 \mid k)\;C \right]^{-1}$$

$$P(k+1 \mid k+1) = P(k+1 \mid k) - \psi(k+1)\;C^T\;P(k+1 \mid k)$$

$$P(k+1 \mid k) = A\;P(k \mid k)\;A^T + \Gamma\;\Gamma^T \qquad (2.2\text{-}2)$$

The notation $x(k+1/k+1)$ has been used to denote the estimate of the state $x(k+1)$ based upon $k+1$ measurements. The quantity $P(k+1/k+1)$ is the covariance of this estimate and corresponds to the matrix P of Section 2.1. From equation (2.1-5), the boundary conditions for the relationships given in (2.2-2) are

$$x(0 \mid 0) = \bar{x}(0)$$

$$P(0 \mid 0) = P_0 \qquad (2.2\text{-}3)$$

This completes the control synthesis for the plant described by equations (2.1-1) and (2.1-2). A block diagram of the complete system is shown in Figure 2-2. The only difference between this system and the optimal regulator problem as formulated by Joseph and Tou lies in the feed-forward term which arises as a result of the deterministic disturbances to the state vector. Computationally the gains K(k) and feedforward term are first computed backward in time using the set of recursive relations given by (2.1-23) and (2.1-24) with the boundary conditions (2.1-25). The estimator processes the available measurement during the control interval and produces new state estimates in accordance with the set of relations given by (2.2-2). The initial values for this set of recursive equations are given by equation (2.2-3).

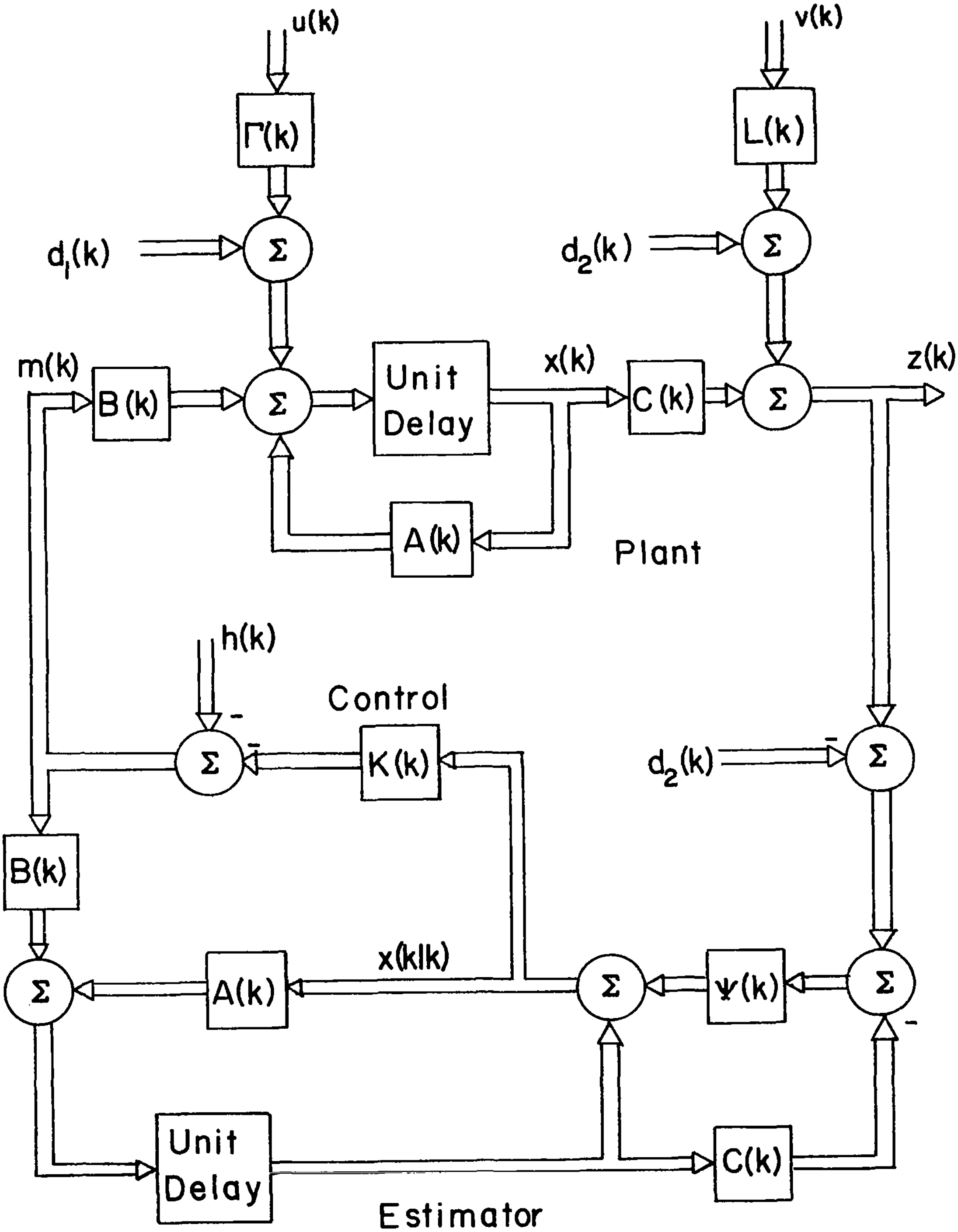

Figure 2.2 Controller Block Diagram

SECTION III

PARAMETER IDENTIFICATION

Having developed the control law for plants with both deterministic and random disturbances plus noisy measurements the problem of identifying the parameters of the plant from these same measurements will be considered. Although there are many methods already in existence, attention will be restricted to those which do not require storage of the entire past measurement history. Specifically, to those capable of on-line implementation. For background, the general requirements of identifiers are first analyzed. The general features of Least Squares techniques are then considered. With this development two methods of identification are analyzed, both of which also generate the state estimates required by the control law of Section II. One of the two is shown to be superior and the manner in which the control is to be altered as a function of the new estimates is considered.

3.1 Identification from Noiseless Measurements

As a preliminary exercise it is instructive to analyze the results obtainable under the assumption that the measurements of the system are perfect. As the simplest case, let the plant be described by a linear, unforced difference equation of order n.

$$x(k + 1) = A\,x\,(k) \tag{3.1-1}$$

If the matrix

$$X(k) = \left[x(k + n - 1) \; . \; . \; . \; x(k) \right] \tag{3.1-2}$$

is defined then, under the assumption that A is constant over the interval $[k, k + n]$, the parameter matrix is given by

$$A = X(k + 1)\, X^{-1}\, (k) \tag{3.1-3}$$

provided the inverse exists. But the matrix X is simply the Wronskian of the system. Therefore the inverse exists, provided all modes of the system are excited. That this should be so seems obvious, intuitively. Notice that the method of identification requires a complete state measurement at each point.

The next case which arises naturally is the forced linear system

$$x(k + 1) = A\,x\,(k) + B\,m(k) \tag{3.1-4}$$

with the control input m(k) having rank r. In line with the philosophy of equation (3.1-2) the matrix quantities

$$M(k) = \left[m(k + n + r - 1) \; . \; . \; . \; m(k) \right],$$

$$X(k) = \left[x(k + n + r - 1) \; . \; . \; . \; x(k) \right] \qquad (3.1\text{-}5)$$

are defined. Then by partitioning of matrices, equation (3.1-4) may be used to obtain A and B according to the equation

$$\left[A \,\vdots\, B \right] = X(k + 1) \left[\frac{X(k)}{M(k)} \right]^{-1} \qquad (3.1\text{-}6)$$

The result again depends upon the existence of an inverse. This time however it is seen that a necessary condition for the inverse to exist is that the m(k)'s not be a linear combination of the x(k)'s over the interval. Thus if a controller of the form

$$m(k) = - K x(k) \qquad (3.1\text{-}7)$$

is used, then for constant gains identification is not possible. This does not mean that if K is a function of time, identification is not possible. Careful reflection will show that this result also is not surprising, since substitution of (3.1-7) into (3.1-4) yields the unforced case of (3.1-1) where

$$x(k + 1) = \left[A - B K \right] x(k) \qquad (3.1\text{-}8)$$

so that, although the matrix $[A - BK]$ is identifiable it is not possible to find A and B separately. It is also concluded from this case that the mode excitation requirement is still present when equation (3.1-7) applies. When the control is not linear, or the gain matrix is time varying, it can be shown[9] that the necessary and sufficient conditions for the system of equation (3.1-4) to be identifiable are that the system be completely controllable and not be a linear combination of the x(k) in the sense discussed above.

Analogous work can be carried out for continuous systems. The essential step in this case is to characterize the continuous time measurements by algebraic quantities. For instance, analogous to (3.1-4) the continuous plant is

$$\frac{dx}{dt} = A\,x(t) + B\,m(t) \qquad (3.1\text{-}9)$$

or

$$x(t) - x(0) = A\int_0^t x(\tau)\,d\tau + B\int_0^t m(\tau)\,d\tau \qquad (3.1\text{-}10)$$

By defining the quantities

$$\int_0^t x(\tau)\,d\tau = z(t), \quad \int_0^t m(\tau)\,d\tau = u(t) \qquad (3.1\text{-}11)$$

equation (3.1-11) is reduced to an algebraic equation. By taking enough different intervals a suitable set of linearly independent equations is obtained as before.

Rather than pursue the problems associated with the identification of systems characterized by (3.1-4) the question of identification when a complete state measurement is not available will be considered. In general, for a linear discrete system, the plant equation has the form

$$
\begin{aligned}
x(k+1) &= A\,x(k) + B\,m(k) \\
y(k) &= C\,x(k) \qquad\qquad (3.1\text{-}12)
\end{aligned}
$$

An immediate problem arises if none of the three matrices A, B and C are known. Essentially the system gain can be distributed between B and C in any fashion and the same transient response from m(k) to y(k) results. Specifically, suppose that M is any non-singular square matrix. Then define a new state variable

$$x^*(k) = M\,x(k) \qquad\qquad (3.1\text{-}13)$$

Equation (3.1-12) may be written as

$$
\begin{aligned}
x^*(k+1) &= M\,A\,M^{-1}\,x^*(k) + M\,B\,m(k) \\
y(k) &= C\,M^{-1}\,x^*(k) \qquad\qquad (3.1\text{-}14)
\end{aligned}
$$

There is no method of discerning equation (3.1-14) from (3.1-12) by observations of m(k) and y(k). For the time being this problem will be put aside by assuming that C is known and the identification problem consists of obtaining A and B from observations of m(k) and y(k).

Another conclusion which may be drawn from the above is that perhaps the matrix formulation is not the best way of approaching the identification problem. For instance suppose a single input, single output system is under discussion. The system is equally well described by the *nth* order difference equation

$$\sum_{i=0}^{N} a_i \, y(k+i) = \sum_{j=0}^{N-1} b_j \, m(k+j) \qquad (3.1\text{-}15)$$

Either one of the other coefficients is known, or a_N may be set to unity without loss of generality. This is analogous to setting

$$C = \begin{bmatrix} 1 & 0 & 0 & . & . & . & 0 \end{bmatrix} \qquad (3.1\text{-}16)$$

in the matrix formulation. Equation (3.1-15) places in evidence the fact that only 2N coefficients need be identified. The discrepancy can be explained by noting that for such a system one matrix formulation is

$$\begin{bmatrix} x_1(k+1) \\ \cdot \\ \cdot \\ \cdot \\ x_N(k+1) \end{bmatrix} = \left[\begin{array}{c|ccc} 0 & & & \\ \cdot & & & \\ \cdot & & I & \\ 0 & & & \\ \hline -a_0 & & & -a_{N-1} \end{array}\right] \begin{bmatrix} x_1(k) \\ \cdot \\ \cdot \\ \cdot \\ x_N(k) \end{bmatrix} + \begin{bmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_N \end{bmatrix} m(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ \\ x_N(k) \end{bmatrix} \qquad (3.1\text{-}17)$$

That is, all of the elements of the A matrix save the last row are specified by the fact that the system is single input, single output.

Conditions for the identification of (3.1-15) may now be easily found. Rewrite it, solving for $y(k + N)$ and re-index the time so that

$$y(k+1) = \sum_{j=0}^{N-1} b_j \, m(k + j - N + 1)$$

$$- \sum_{i=0}^{N-1} a_i \, y(k + i - N + 1) \qquad (3.1\text{-}18)$$

or

$$y(k+1) = z^T(k)\,\alpha = \Big[y(k), \; . \; . \; . \; y(k-N+1),$$

$$m(k), \; . \; . \; . \; m(k-N+1)\Big] \begin{bmatrix} -a_{N-1} \\ \cdot \\ \cdot \\ \cdot \\ -a_0 \\ b_{N-1} \\ \cdot \\ \cdot \\ \cdot \\ b_0 \end{bmatrix} \tag{3.1-19}$$

In equation (3.1-19) the unknown parameters are all contained in the vector α, and the past measurement history in $z(k)$. If 2N measurements are taken then the total set of measurements and their relations to the parameter vector α may be written as

$$\begin{bmatrix} y(k+1) \\ \\ y(k+2N) \end{bmatrix} = \begin{bmatrix} y(k) \; . \; . \; . \; y(k-N+1), \; m(k) \; . \; . \; . \; m(k-N+1) \\ \vdots \\ y(k+2N-1) \ldots y(k+N), \; m(k+2N-1) \ldots m(k+N) \end{bmatrix} \alpha \tag{3.1-20}$$

or $$y = Z\,\alpha$$

Therefore $$\alpha = Z^{-1}\,y \tag{3.1-21}$$

and the identifiability of this system rests upon the existence of the inverse in (3.1-21). However the matrix Z is seen to consist of the forced responses of the system. Clearly, a necessary condition for the inverse to exist requires that the control m(k) not be a constant linear function of y over the identification interval. It can be shown[19] that all single input single output systems are reducible to the form given in equation (3.1-15) so that these results hold generally.

Lee[19] used this formulation as the starting point for applying Least Squares filtering. However, it is not necessary to restrict the discussion to single input, single output systems, just as Least Squares filtering is not restricted to scalar measurements. Consider again the vector system of equations (3.1-12). In terms of z transfer functions they may be written

$$y(z) = C \left[z\,I - A \right]^{-1} B\, m(z) \qquad (3.1\text{-}22)$$

The inverse in (3.1-22) always exists. Equation (3.1-22) represents the z transform of a set of difference equations. It is equivalent to the set of equations

$$\sum_{i,\,j} a_{ij}\, y_j = \sum_{i\,k} b_{ik}\, m_k \qquad (3.1\text{-}23)$$

Since the original matrix equation was nth order there can effectively be only n a coefficients which are non-zero in the set (3.1-23). Assuming, as before, that the C matrix is known, then knowing the a coefficients is equivalent to knowing the A matrix. As a result only the B matrix is unknown. This has at most n x r coefficients, where r is the number of inputs. Consequently an nth order system with r inputs is specified at the minimum by n(r + 1) coefficients.

The above discussion points out some salient features of the identification problem. Even with complete state measurement all modes of the system must be excited. Moreover, if the system is being controlled during the identification interval the controller cannot be composed of constant feedback gains. In the event that the states are not observable directly, it is further necessary that the system be observable. This requirement is also seen to be a natural consequence since a non-observable mode could not, by definition, be measured.

All of the preceding material has assumed constant coefficients and precise measurements. Suppose one of the identification techniques is applied twice, at two different time intervals, and the results differ. Which one is right? Is it the result of a time variation of the a's or the result of measurement noise? Or possibly the plant is only approximated by linear differential equations. Considerations such as these lead naturally to some method of continual observation and updating of the identification. If one of the methods just discussed is to be employed then some method must be found to accommodate the new data and include it with the old. But this is precisely the problem for which the classical "Least Squares estimation" is used. Hence the next section is devoted to a discussion of this technique for identifying systems.

3.2 Least Squares Estimation

For the sake of unity in the presentation, this Section is included although the results are identical to those of Section 2.2. For later convenience, those results will now be derived using a maximum likelihood approach. The general problem to be considered is as follows. Given a dynamic system

$$x(k + 1) = A\,x(k) + B\,m(k) + \Gamma\,u(k) \tag{3.2-1}$$

and

$$y(k) = C\,x(k) + L\,v(k)$$

where only the y's are measurable, and they are corrupted by noise, $v(k)$. The state $x(k)$ is also probabilistic due to the random input signal u(k). The pertinent statistics of u and v are

$$E\{u(k)\} = 0 \qquad E\{v(k)\} = 0$$

$$E\{u(i)\ u^T(j)\} = \delta_{ij}\ ;\ E\{v(i)\ v^T(j)\} = \delta_{ij}$$

$$E\{v(i)\ u^T(j)\} = 0 \tag{3.2-2}$$

(δ_{ij} is the Kronecker delta).

This is the discrete analog of the problem considered by Bryson and Frazier[5]. Suppose for the moment that x is not a state variable, but a constant vector. Then the problem reduces to making successive measurements.

$$z(k) = C\ x(k) + L\ v(k) \tag{3.2-3}$$

and fitting an estimate $\hat{x}$ to these measurements. It is not necessary to restrict the discussion to scalar measurements, so that z(k) may be a vector. As a notational convenience the estimate of x based upon i measurements will be written $\hat{x}_i$. Using a maximum likelihood approach the likelihood function

$$L(z, x) = p(z|x)$$

is set up with z denoting the complete set of measurements as before. Since (3.2-3) is linear the likelihood function may be manipulated using Bayes Rule so that

$$L(z, x) = p(z|x) = \frac{p(z, x)}{p(x)} = \frac{p(L\ v)\ p(x)}{p(x)} = p(L\ v) \tag{3.2-4}$$

The notation $p(L\ v)$ refers to the density function of the random variable $L\ v$. Now if v is Gaussian so is y and therefore the likelihood function is

$$L(z, x) = \frac{1}{(2\pi)^{\frac{kr}{2}}\ |LL^T|}$$

$$\exp\left[-\frac{1}{2}\sum_{i=1}^{k}(z_i - C\,x)^T\,(LL^T)^{-1}\,(z_i - C\,x)\right] \tag{3.2-5}$$

(r denotes the rank of z, the measurement). Maximizing the likelihood function with respect to x is equivalent to minimizing the exponent. The sum may be written as the vector equation

$$Z(k) = \begin{bmatrix} z(1) \\ \\ z(k) \end{bmatrix} = \xi_k\ x = \begin{bmatrix} C^T(1) \\ \\ C^T(k) \end{bmatrix} x \tag{3.2-6}$$

So that the maximum likelihood estimate is that x which minimizes

$$J = \frac{1}{2}\left(Z(k) - \xi_k\,x\right)^T (LL^T)^{-1}\left(Z(k) - \xi_k\,x\right) \tag{3.2-7}$$

The minimum is found, by setting $\frac{\partial J}{\partial x} = 0$, to be

$$\hat{x}_k = \left[\xi_k{}^T (LL^T)^{-1} \xi_k \right]^{-1} \xi^T (LL^T)^{-1} Z(k) \qquad (3.2\text{-}8)$$

The estimate $\hat{x}_k$ is unbiased, since

$$E\left[\hat{x}_k\right] = \left[\xi_k{}^T (LL^T)^{-1} \xi_k \right]^{-1} \xi^T (LL^T)^{-1} E\left[Z(k)\right]$$

and

$$E\left[z(k)\right] = \xi\, x \qquad (3.2\text{-}9)$$

The covariance of the estimate is easily shown to be

$$\text{cov } \hat{x}_k = \left[\xi_k{}^T (LL^T)^{-1} \xi_k \right]^{-1} \qquad (3.2\text{-}10)$$

The question of one additional measurement $z(k + 1)$ and its effect upon the estimate is now considered. The new estimate, denoted $\hat{x}_{k+1}$, is immediately written down by inspection of equation (3.2-8) as

$$\hat{x}_{k+1} = \left[\xi^T_{k+1} (LL^T)^{-1} \xi_{k+1} \right]$$

$$\xi^T_{k+1} (LL^T)^{-1} Z(k + 1) \qquad (3.2\text{-}11)$$

But by noting that ξ_{k+1} and $Z(k + 1)$ can be partitioned into

$$\xi_{k+1} = \begin{bmatrix} \xi_k \\ \hline C \end{bmatrix}, \quad Z(k+1) = \begin{bmatrix} Z(k) \\ \hline z(k+1) \end{bmatrix} \tag{3.2-12}$$

Equation (3.2-11) may then be written as

$$\hat{x}_{k+1} = \left[\xi_k^T (LL^T)^{-1} \xi_k + C^T(LL^T)^{-1} C\right]^{-1}$$

$$\left[\xi_k^T (LL^T)^{-1} Z(k) + C^T(LL^T)^{-1} z(k+1)\right] \tag{3.2-13}$$

Now the inverse term is, by comparison to (3.2-10), the covariance of the estimate $\hat{x}_{k+1}$. At this point a convenient matrix relation must be introduced.

Lemma: If the inverse of a nonsingular matrix A is given by

$$A^{-1} = B^{-1} + C(LL^T)^{-1} C^T \tag{3.2-14}$$

then the matrix A itself is equal to

$$A = B - BC\left[C B C^T + LL^T\right]^{-1} C^T B \tag{3.2-15}$$

provided that the inverse of B and LL^T exist.

Proof: The inverse of the matrix A is that matrix for which

$$AA^{-1} = A^{-1} A = I \tag{3.2-16}$$

By direct matrix multiplication of equations (3.2-14) and (3.2-15) in either order equation (3.2-16) is obtained.

Using these identities and denoting the covariance of $\hat{x}_k$ by P_k equation (3.2-13) may be shown to become

$$\hat{x}_{k+1} = \hat{x}_k + P_k C^T \left[C P_k C^T + LL^T \right]^{-1} \left[z(k+1) - C \hat{x}_k \right]$$

(3.2-17)

and the covariance of the new estimate is related to that of the old estimate by the relation

$$P_{k+1} = P_k - P_k C^T \left[C P_k C^T + LL^T \right]^{-1} C P_k$$

(3.2-18)

These equations show how, given an unbiased estimate $\hat{x}$ and the covariance of that estimate, a new measurement with a known error (Lv) is incorporated to update the estimate and reduce the covariance. Indeed the same result can be achieved more rapidly from this viewpoint using maximum likelihood. Recall that the likelihood function was seen to be

$$L(z, x) = p(z|x) = p(v) = p(v_1)\, p(v_2) \cdots p(v_k)$$

(3.2-19)

But the errors of all terms save the last are incorporated in the estimate $\hat{x}_k$ and the covariance P_k , therefore the maximum likelihood is equivalently obtained by minimizing the functional

$$J = \frac{1}{2}\left((x_k - x_{k+1})^T P_k^{-1}(\hat{x}_k - \hat{x}_{k+1}) + \left[z(k+1) - C\hat{x}_{k+1}\right]^T (LL^T)^{-1} \left[z(k+1 - C\hat{x}_{k+1}\right]\right) \tag{3.2-20}$$

with respect to $\hat{x}_{k+1}$. The result by ordinary calculus and the matrix inversion lemma is

$$\hat{x}_{k+1} = \hat{x}_k + P_k C^T \left[C P_k C^T + LL^T\right]^{-1} \left[z(k+1) - C x_k\right] \tag{3.2-21}$$

If $\hat{x}_k$ is un unbiased estimator then so is $\hat{x}_{k+1}$ since

$$E\{\hat{x}_{k+1}\} = \left[P_k^{-1} + C^T(LL^T)^{-1} C\right]^{-1} \left[P_k E\{\hat{x}_k\} + C^T(LL^T)^{-1} E\{x\}\right] \tag{3.2-22}$$

Finally, the covariance of the new estimate is found by substitution into (3.2-20) and use of the inversion lemma to be

$$P_{k+1} = P_k - P_k C^T \left[C P_k C^T + LL^T \right]^{-1} C P_k$$

(3.2-23)

which is exactly the same as the previous result.

With this result the original problem posed in equations (3.2-1) and (3.2-2) can be easily analyzed. Suppose after some time an estimate $\hat{x}$ of the state at time k is obtained based upon observations up to and including time k. This estimate is denoted $\hat{x}(k|k)$ and it is further assumed that the covariance of the estimate is known and denoted by $P(k|k)$. Now, one time interval later, the true state has changed from $x(k)$ to $x(k+1)$ according to equation (3.2-1). The best prediction of $x(k+1)$ based upon the measurements up to time k is the linear extrapolation

$$\hat{x}(k+1|k) = A\, x(k|k) + B\, m(k) \qquad (3.2\text{-}24)$$

This result has been shown by many authors dating back to Wiener[28]. The estimate is unbiased if $\hat{x}(k|k)$ is unbiased and the covariance is given by the propagation of uncertainty in a linear dynamic system.

$$\operatorname{cov}\{\hat{x}(k+1|k)\} = A\, P(k|k)\, A^T + \Gamma\Gamma^T = P(k+1|k)$$

(3.2-25)

But now the problem is reduced to the one previously discussed, in that an old estimate $\hat{x}(k+1|k)$ of the state $x(k+1)$ and its covariance is known from equations (3.2-24) and (3.2-25). A new measurement $z(k+1)$ is made and equations (3.2-21) and (3.2-23) prescribe the manner in which the new measurement alters the estimate and the covariance. Using the notation $P(k+1|k+1)$ for the new covariance and $\hat{x}(k+1|k+1)$ for the new estimate (3.2-21) and (3.2-23) may be written as

$$\hat{x}(k+1|k+1) = \hat{x}(k+1|k) + \psi(k+1)\left[z(k+1) - C\hat{x}(k+1|k)\right]$$

$$P(k+1|k+1) = P(k+1|k) - \psi(k+1)\, C\, P(k+1|k)$$

$$\psi(k+1) = P(k+1|k)\, C^T\left[C\, P(k+1|k)\, C^T + LL^T\right]^{-1} \tag{3.2-26}$$

The two quantities $P(k+1|k)$ and $\hat{x}(k+1|k)$ being given by the previous two equations. These are exactly the results given in Section 2.2. On the other hand this derivation has a certain appeal in that it places in evidence the reason for the structure of the equations whereas Kalman's use of orthogonal projections tends to obscure the physical reasoning behind the equations.

3.3 Identification Using Least Squares

It would appear from the material just developed that the problem is solved. All discrete systems may be expressed as a set of difference equations. For simplicity, the single input, single output system will be discussed. As shown in Section 3.1 the system difference equation may be written in the form

$$y(k + 1) = Y^T\alpha = \sum a_i \; y(k - i) + \sum b_j \; m(k - j) \tag{3.3-1}$$

The vector y is then seen to play the role of the matrix C in the least squares development and by comparison to equation (3.2-3) the coincidence is immediately seen. The question which must be answered now is: by what mechanism does the noise enter into the system? Lee restricted himself to input noise for reasons which will soon be apparent. If the noise is all at the input then the form of the state equation is

$$x(k + 1) = A\,x(k) + B\,m(k) + \Gamma\,u(k)$$

$$y(k) = C\,x(k) \tag{3.3-2}$$

where

$$A = \begin{bmatrix} 0 & & \\ \cdot & & \\ \cdot & I & \\ \cdot & & \\ 0 & & \\ \hline a_1 & & a_N \end{bmatrix} \qquad B^T = \begin{bmatrix} b_1 & \cdot & \cdot & \cdot & b_N \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & \cdot & \cdot & 0 \end{bmatrix}$$

The first thing to be noticed is that it will be necessary to know the parameters in the vector Γ . This is equivalent to knowing the covariance of the noise. By a standard transformation[7] the matrix equation (3.3-2) may be written in the form of a first order difference equation

$$\begin{aligned} y(k + N) - a_N\, y(k + N - 1) - \ldots - a_1\, y(k) \\ = b_N'\, m(k + N - 1) + \ldots + b_1'\, m(k) \\ + \gamma_N'\, u(k + N - 1) + \ldots + \gamma_1'\, u(k) \end{aligned} \qquad (3.3\text{-}3)$$

The b and γ coefficients have been primed to indicate that they are not the same as the coefficients of the vectors B and Γ but are linearly related to them and the matrix A. Now, equation (3.3-3) appears to be in the proper form for the application of the previously developed Least Squares technique if the $\gamma_i u_j$ terms are lumped together as one total disturbance so that equation (3.3-3) is written

$$y(k+1) = \sum_{i=0}^{N-1} \left[a_{i+1}\, y(k-i) + b_{i+1}\, m(k-i) \right] + \text{dist.}$$

$$= Y^T(k)\, \alpha + \text{dist.} \qquad (3.3\text{-}4)$$

However, although equation (3.3-4) is of the same form as, say (3.2-3), there is one difference. The disturbance term is correlated whereas in the Least Squares work this was assumed not to be the case. To see this, write out the disturbance at two successive times

$$\text{dist}\ (k) = \gamma_N'\, u(k) + \gamma_{N-1}'\, u(k-1) + \ldots + \gamma_1'\, u(k-N+1)$$

$$\text{dist}\ (k+1) = \gamma_N'\, u(k+1) + \gamma_{N-1}'\, u(k) + \ldots + \gamma_1'\, u(k-N+2)$$

$$(3.3\text{-}5)$$

Hence

$$E\ \{\text{dist}\ (k)\ \text{dist}\ (k+1)\} = (\gamma_N\, \gamma_{N-1} + \ldots + \gamma_2\, \gamma_1)\ E\ \{u^2\}$$

$$(3.3\text{-}6)$$

Therefore the Least Squares estimation is not optimal for this case. There are two ways around this dilemma. The first method is to return to the formulation given by (3.3-2). If no updating is made until N outputs are obtained then essentially a complete state measurement has been made and (3.3-4) may be rewritten as

$$x(k+1) = \begin{bmatrix} y(k+1) \\ \vdots \\ y(k+N) \end{bmatrix} = \begin{bmatrix} Y^T(k) \\ \\ Y^T(k+N-1) \end{bmatrix} \alpha + \Gamma\, u(k) \tag{3.3-7}$$

That is, an entire measurement of the state at time k is possible, but to obtain it the outputs must be measured over an interval of N samples. After the N sample, wait a new measurement of the complete state that obtained at time k is available. The disturbance u in this case is not correlated since it may be treated as a vector. The other method of analytically handling this difficulty is to adjoin additional states to the system description which account for the correlated noise. This method adds appreciably to the system complexity.

Rather than pursue this topic, a second and more serious restriction will now be considered. Suppose that the measurements of the output are not precise due to additive noise. That is the actual measurements z(k) are related to the state by

$$z(k) = C\, x(k) + L\, v(k) \tag{3.3-8}$$

where v(k) is a white Gaussian random noise of zero mean and unit covariance. Then, although equation (3.3-4) has the appearance of being in the proper form for a Least Squares technique, it is not. The measurement matrix y^T is itself composed of random variables since each y is in fact

$$y^T(k) = \Big[z(k),\ z(k-1)\ .\ .\ .\ z(k-N+1),$$

$$m(k)\ .\ .\ .\ m(k-N+1)\Big] \qquad (3.3\text{-}9)$$

and the z's are as given by equation (3.3-8). As a result there can be no guarantee of covergence to the correct result, even when the parameter vector α is time invariant.

A heuristic method of avoiding this problem is to regard the unknown parameters as being themselves states which are driven by noise. This prevents the estimated covariance from decreasing monotonically to zero as evidenced by (3.2-25) and (3.2-26). Alternatively, recall that the control for a system with noisy measurements requires a Wiener-Kalman filter to estimate the state given the measurement. If this estimate represents the "best" estimate of the state then it could be used in place of the noisy measurements in estimating the parameters. The rub of course is that the state estimate is made assuming the parameters are known as evidenced by (3.2-24) which gives the predicted state based upon the old state estimate and the parameters. However, it is a straightforward step to reason that since both a state and a parameter estimate are needed they be done as a two step recursive process.

1. Estimate the state using old parameters.

2. Estimate the new parameters using the new state estimate.

3. Repeat 1 and 2 with new data.

The state estimator equations are as given by (2.2-2), except that the A, B and C matrices must be regarded as estimates of the parameters rather than the parameters themselves. It is also necessary to exercise care about the form of the difference equations in each case. For the Least Squares technique a set of first order difference equations is used. However, for the parameter identification a difference equation of the type typified by (3.3-1) is required. A suitable transformation is available for the single input, single output case which reconciles the two representations[19]. Essentially, the A matrix must be put in the form of (3.3-2). Even then the correspondence is not complete unless only one element of the B matrix is nonzero. Otherwise the elements b' of equation (3.3-3) are related to the elements of the B matrix by the set of equations[7].

$$
\begin{aligned}
b'_N &= b_1 \\
b'_{N-1} &= b_2 - a_N b_1 \\
&\vdots \\
b'_1 &= b_N - a_N b_{N-1} - \cdots - a_2 b_1
\end{aligned}
\qquad (3.3\text{-}10)
$$

Equation (3.3-10) shows that the transformation involves the a's, some or all of which may be estimates. Alternatively, depending upon the mechanism by which noise is regarded as forcing the system, the estimate $\hat{x}(k+1|k)$ could be generated using (3.3-3) rather than (3.3-2). In either event, the important point is that there exists a disparity in the two formulations required respectively by the state identifier and the parameter identifier.

In the case of multiple outputs the problem is even more aggravated. For two outputs a transformation of the type given by (3.3-10) cannot be written unless they are essentially position and rate type measurements of the same system. Moreover, what of the control which must be updated using the new parameter estimates? If optimal control is to be used then the weighting matrices Q and S which are used are based upon a specific definition of the state variables. Therefore, a second transformation may well be required whenever the control is updated. One possible method would be to transform Q and S, based upon the nominal or a priori estimates of the parameters, to the state definition used by the estimator and use these values for Q and S.

These are primarily computational difficulties, in the use of the proposed two step procedure. To summarize, assume a single input-single output system. It is further assumed that the state description is also that used in formulating the control weighting matrices Q and S. The plant is then described by the difference equation

$$y(k + N) - a_N\, y(k + N - 1) \; . \; . \; . - a_1\, y(k)$$

$$= b_N'\, m(k + N - 1) \; . \; . \; . + b_1'\, m(k) \qquad (3.3\text{-}11)$$

The measurable output z(k) is corrupted by additive noise

$$z(k) = y(k) + L\, v(k) \qquad (3.3\text{-}12)$$

where v(k) is a white sequence of zero mean and unit variance. Noise is present in the states as described by the equivalent matrix formulation of equation (3.3-2).

The parameters b_1 to b_N in the B vector are related to the parameters b_N' to b_1' in equation (3.3-11) according to the transformation set forth by (3.3-10). The equivalent noise input gain of u(k) into equation (3.3-11) is computed using the same set of relations (3.3-10) but now the matrix Γ plays the role of the B matrix. As was remarked previously if the formulation of equation (3.3-11) is used then the input noise is correlated. On the other hand, to avoid this requires updating the parameters only every N samples. In either event, the state estimate is made from

$$\hat{x}(k+1|k+1) = \hat{x}(k+1|k) + \psi_x(k+1)\left[z(k+1) - C\hat{x}(k+1|k)\right]$$

$$\hat{x}(k+1|k) = \hat{A}(k)\,\hat{x}(k|k) + \hat{B}(k)\,m(k)$$

$$\psi_x(k+1) = P_x(k+1|k)\,C^T\left[C\,P_x(k+1|k)\,C^T + LL^T\right]^{-1}$$

$$P_x(k+1|k+1) = P_x(k+1|k) - \psi_x(k+1)\,C\,P_x(k+1|k)$$

$$P_x(k+1|k) = \hat{A}(k)\,P_x(k|k)\,\hat{A}^T(k) + \Gamma_x\,\Gamma_x^T \tag{3.3-13}$$

The notation $\hat{A}$ and $\hat{B}$ has been employed to denote the fact that the parameters used are estimates, the true parameters not being known. The covariances are noted as belonging to the state estimate through the use of a subscripted x. The parameters are estimated via Least Squares smoothing of equation (3.3-11). Variations in the parameters with time may be accounted for by assuming a difference equation for the parameters of the form

$$a_i(k + 1) = a_i(k) + \gamma_{pi}\, u(k) \tag{3.3-14}$$

If the correlations in the input noise are ignored then the parameters are estimated by the equations

$$\hat{\alpha}(k+1|k+1) = \hat{\alpha}(k) + \psi_p(k+1)\left[z(k+1) - y^T(k)\hat{\alpha}(k)\right]$$

$$\psi_p(k+1) = P_p(k+1|k)\; y(k)\left[y^T(k)\; P_p(k+1|k)\, y(k) + L_p\, L_p^T\right]^{-1}$$

$$P_p(k+1|k+1) = P_p(k+1|k) - \psi_p(k+1)\; y^T\; P_p(k+1|k)$$

$$P_p(k+1|k) = P_p(k|k) + \Gamma_p\, \Gamma_p^T \tag{3.3-15}$$

and the vectors $Y(k)$ and $\alpha(k)$ are

$$Y^T(k) = \left[\hat{x}(k) \;\ldots\; \hat{x}(k-N+1),\; m(k) \;\ldots\; m(k-N+1)\right] \tag{3.3-16}$$

$$\hat{\alpha}(k) = \left[\hat{a}_N(k)\ldots\; \hat{a}_1(k),\; \hat{b}_N'(k)\;\ldots\; \hat{b}_1'(k)\right] \tag{3.3-17}$$

In the event that the noise correlation is taken into account so that parameter updating is performed only once every N samples the parameter estimate is given by

$$z(k + 1) = y^T(k) + L_p\, u_k$$

$$z(k + N - 1) = y^T(k + N) + L_p\, u_{k + N}$$

(3.3-18)

The term $L_p\, u_k$ represents the complete sum of the measurement noise $L\, v(k + 1)$ and the input noise over the previous interval so that it may be written

$$L_p\, u_k = L\, v(k + 1) + \gamma_N'\, u(k) + \gamma_{N-1}'\, u(k - 1)$$

$$+ = + \gamma_1\, u(k - N) \qquad (3.3\text{-}19)$$

The γ's being those referred to in equation (3.3-3) and related to the Γ matrix by the customary transformation.

This then is the sequential estimation of the parameters and states. It is well to recall that this method was advanced because when Least Squares techniques were considered for parameter estimation it was noted that the measurement matrix involved was itself noisy. Now however, the use of the $\hat{A}$ and $\hat{B}$ notation shows that the same thing applies in the state estimation. The essential difficulty of course is that the estimate, $x(k+1|k)$, is dependent upon both the parameter and state estimates and these two are not independent. The sequential estimation scheme takes no account of the correlation between the state and the parameter estimate.

3.4 Simultaneous Estimation

The problem is now reduced to one of accounting for the mutual dependence of the estimated states and parameters. A method frequently suggested in the literature is to adjoin the parameters to the system of state equations, regarding the parameters as added state variables. However, this effectively replaces the linear plant description with a nonlinear one since when the parameters are treated as additional states the terms of the old state equations are composed of products of the parameters (new states) and the actual states. Although nonlinear estimation is not unknown, it usually involves retaining the entire measurement history and fitting the function to that. On the other hand, if the identification is to be done "on line" then such a method is clearly undesirable. Indeed, the attractive feature of the Least Squares techniques which have been considered so far is the recursive form of the computations.

That is, the entire past is contained in the present estimate and present covariance. What is sought then is a method with the computational simplicity of Least Squares smoothing which can handle nonlinear systems.

To be specific consider the nonlinear discrete system

$$x(k+1) = f\big(x(k),\ m(k),\ u(k),\ k\big) \tag{3.4-1}$$

The new state $x(k+1)$ is a function of the old state, a deterministic input and a disturbance input $u(k)$, representing all statistical inputs. The mean of u is taken as zero without loss of generality. The problem is to estimate the state based upon noisy measurements of the system. The measurements are assumed to be linear functions of the states. In equation form

$$z(k) = C(k)\ x(k) + L(k)\ v(k) \tag{3.4-2}$$

The noise vector $v(k)$ is assumed to be white Gaussian with mean zero and unit variance. All matrices may vary in time, as denoted by their arguments.

Suppose that at time k, by some method or other, an estimate of the state were available together with the variance of this estimate. Denote the estimate $\hat{x}(k|k)$ and the covariance matrix $P(k|k)$, indicating the estimate is based upon all measurements up to the present time. The error between the estimate and the actual state is

$$\delta\hat{x}(k|k) = x(k) - \hat{x}(k|k) \tag{3.4-3}$$

Now, one sample time later the state evolves in accordance with equation (3.4-1). Equation (3.4-1) is expanded in a Taylor Series about $\hat{x}(k|k)$ and $u(k) = 0$.

$$x(k + 1) = f(\hat{x}, m, u, k) + \left.\frac{\partial f}{\partial x}\right|_{\hat{x}} \delta\hat{x}(k|k)$$

$$+ \left.\frac{\partial f}{\partial u}\right|_{u=0} u(k) + \ldots \tag{3.4-4}$$

But, the estimate of $x(k + 1)$ conditioned on measurements up to time k must be

$$\hat{x}(k + 1|k) = f\Big(\hat{x}(k|k), m(k), 0, k\Big) \tag{3.4-5}$$

Thus the error at time k + 1 conditioned upon k measurements must be

$$\delta\hat{x}(k + 1|k) = x(k + 1) - \hat{x}(k + 1|k) \tag{3.4-6}$$

Combining equations (3.4-4) through (3.4-6) gives the equation for the error as

$$\delta\hat{x}(k + 1|k) = \left.\frac{\partial f}{\partial x}\right|_{\hat{x}(k|k)} \delta\hat{x}(k|k) + \left.\frac{\partial f}{\partial u}\right|_{u=0} u(k) + \ldots \tag{3.4-7}$$

If the higher order terms of (3.4-7) are ignored then a linear equation is obtained for the error propagation from time k to time k + 1. The covariance of $\delta\hat{x}(k + 1|k)$ is then easily found in exactly the same manner as in Section 3.2 and is

$$P(k + 1|k) = \left(\frac{\partial f}{\partial \hat{x}}\right) P(k|k) \left(\frac{\partial f}{\partial \hat{x}}\right)^T + \left(\frac{\partial f}{\partial u}\right)\left(\frac{\partial f}{\partial u}\right)^T \tag{3.4-8}$$

Now, a new measurement is made of the actual state x(k + 1). But by the methods developed in Section 3.2 it is a simple matter to combine these two estimates of x(k + 1). The old estimate is $\hat{x}(k + 1|k)$. The covariance of this estimate is given by equation (3.4-8). The new estimate, based upon the new measurement, is

$$\hat{x}(k+1|k+1) = \hat{x}(k+1|k) + \psi(k+1)\left[z(k+1) - C\hat{x}(k+1|k)\right] \tag{3.4-9}$$

The correction matrix $\psi(k + 1)$ is given, as usual, by the recursive equations

$$\psi(k + 1) = \left(\frac{\partial f}{\partial \hat{x}}\right) C^T \left[C\left(\frac{\partial f}{\partial \hat{x}}\right) C^T + LL^T\right]^{-1} \tag{3.4-10}$$

The covariance of the new estimate is

$$P(k + 1|k + 1) = P(k + 1|k) - \psi(k + 1)\ C\ P(k + 1|k) \tag{3.4-11}$$

Equations (3.4-9) through (3.4-11) then show how a new estimate and its covariance are obtained from the old estimate and covariance. Equation (3.4-5) prescribes how the conditional estimate of $x(k + 1)$ is made and (3.4-8) gives its covariance. These equations therefore constitute the desired estimator for nonlinear systems using the recursive computational techniques of Least Squares filtering.

The validity of the estimation procedure above depends of course upon the effect of the higher order terms in equation (3.4-7) which were conveniently dropped. By restricting the discussion to the identification of linear systems some general results may be obtained. Consider then the familiar linear system of Nth order

$$x(k + 1) = A(k)\ x(k) + B(k)\ m(k) + \Gamma(k)\ u(k)$$

$$z(k) = C(k)\ x(k) + L(k)\ v(k) \tag{3.4-12}$$

It is assumed that certain, and possibly all, of the parameters in the A matrix are to be estimated. An exactly similar procedure may be used to estimate unknown means in the disturbance $u(k)$ or terms in the matrix B. These unknown parameters shall be regarded as additional states. To account for possible unknown time variations they will be regarded as having dynamical equations of the form

$$a_i(k + 1) = a_i(k) + \gamma_{pi}\, u_p(k) \tag{3.4-13}$$

The $u_p(k)$ representing a zero mean unit variance noise. If some a priori knowledge of the manner in which the parameters are varying in time is available this may be included by rewriting equation (3.4-13) in the form

$$a_i(k + 1) = w(k)\, a_i(k) + \left[1 - w(k)\right] \bar{a}_i(k + 1) + \gamma_{pi}\, u_p(k) \tag{3.4-14}$$

The term $\bar{a}_i(k + 1)$ represents an expected value or nominal value for the parameter and $w(k)$ is a weighting factor between the prediction and the nominal parameter. Equation (3.4-14) equivalently states that the parameters drift back to the nominal unless driven off by the fictitious noise which provides the required statistical representation for the parameter variations. Symbolically, the set of equations (3.4-14) may be written as a matrix equation.

$$a(k+1) = W(k)\, a(k) + \left[I - W(k)\right] \bar{a}(k+1) + \Gamma\, u_p(k) \tag{3.4-15}$$

The matrix w is diagonal and composed of the elements w (k). The transition equation (3.4-15) is now added on to the transition equation in (3.4-12) giving an augmented system equation

$$x_A(k+1) = A_A(k)\ x_A(k) + B_A(k)\ m_A(k) + \Gamma_A(k)\ u_A(k)$$

$$z(k) = C_A(k)\ x_A(k) + L\ v(k) \qquad (3.4\text{-}16)$$

where the matrices are

$$A_A = \begin{bmatrix} A & 0 \\ 0 & W \end{bmatrix}, \quad B_A = \begin{bmatrix} B \\ \hline I - W \end{bmatrix}, \quad \Gamma_A = \begin{bmatrix} \Gamma \\ \Gamma_P \end{bmatrix}$$

$$C_A = \left[\begin{array}{c|c} C & 0 \end{array}\right]$$

and the state and input vectors are

$$x_A = \begin{bmatrix} x \\ \hline a \end{bmatrix} \qquad m_A = \begin{bmatrix} m(k+1) \\ \hline \bar{a}(k+1) \end{bmatrix}$$

It should be remembered that certain of the coefficients in the A matrix of equation (3.4-16) are the added states and therefore, although the equation appears linear, it is not.

To see more clearly what is going on, consider a representative scalar equation from (3.4-16).

$$x_j(k+1) = a_{ij}\, x_1(k) + \ldots + a_{Nj}\, x_N(k) + b_j\, m(k) \tag{3.4-17}$$

Then the first order Taylor Series expansion of (3.4-17) is

$$\delta x_j(k+1) = \hat{a}_{ij}\, \delta x_1(k) + \ldots + \hat{a}_{Nj}\, \delta x_N(k) + \hat{x}_1(k)\, \delta a_{ij} + \ldots + \hat{x}_N(k)\, \delta a_{Nj} \tag{3.4-18}$$

Hence the general form of the linear expansion is

$$\left.\frac{\partial f}{\partial x}\right|_{\hat{x}} = \left[\begin{array}{c|c} \hat{A}(k) & \hat{X}(k) \\ \hline 0 & I \end{array}\right] \tag{3.4-19}$$

and similarly

$$\left.\frac{\partial f}{\partial u}\right|_{u=0} = \Gamma_A \tag{3.4-20}$$

Now, because the linear system with augmented states is bilinear, all of the higher order terms beyond second are identically zero. The only second order terms which arise in the expansion are crossproducts of the form $\delta a\, \delta x$. Consider a second order system

$$x_2(k+1) = a_1\, x_1(k) + a_2\, x_2(k) \tag{3.4-21}$$

The complete expansion is

$$\delta x_2(k + 1) = \hat{a}_1 \delta x_1 + \hat{a}_2 \delta x_2 + \hat{x}_1 \delta a_1 + \hat{x}_2 \delta a_2$$

$$+ \delta a_1 \delta x_1 + \delta a_2 \delta x_2 \qquad (3.4\text{-}22)$$

Equation (3.4-22) places in evidence the fact that sufficient conditions for the second order terms to be small are

$$\delta x << \hat{x}, \ \delta a << \hat{a} \qquad (3.4\text{-}23)$$

But this is equivalent to asking for small percentage errors in the estimates. Thus, whenever the estimates at time k are "good" the above procedure can accurately track the state to time k + 1.

It is now evident from inspection of (3.4-9) and (3.4-10) why this system is to be preferred over the sequential estimation technique discussed in the previous Section. The matrix $P(k + 1|k)$ of the covariance specifically accounts for the correlation between the states and the parameters. In fact, from the partitioned matrix formulation of equation (3.4-19) it is clear that the sequential estimation method not only partitioned the matrices as shown in (3.4-16) but also assumed the covariance to be in the form

$$P(k|k) = \left[\begin{array}{c|c} P_x(k|k) & 0 \\ \hline 0 & P_p(k|k) \end{array}\right] \qquad (3.4\text{-}24)$$

Specifically, the two step procedure ignores the correlation between the state and parameter estimates. An added advantage of this technique is the simultaneous generation of the state and parameter estimates without the necessity for linear transformations and two disparate state representations. Moreover, it is not dependent upon a specific form of the A matrix as was the sequential estimation scheme. Thus the form used for the state representation should be the same as that used in computing the control thus avoiding all state transformations.

Rather than give examples of the use of this technique here it will be illustrated through application to the problem of controlling a large flexible launch vehicle which is discussed in Section IV. Although Section III has been discussing identification problems it is important that they be considered in conjunction with the control system and the method of synthesis used for the controller. Consequently the next section discusses briefly how the new parameter estimates might be used to alter the control loop.

3.5 Control Using New Parameter Estimates

The rationale for employing optimal control techniques in SECTION II was their great generality. The price paid for this powerful tool is the celebrated two-point boundary value problem. For linear plants the problem is separable and the feedback gains may be computed from a difference equation. However, the difference equation runs backward in time having its initial conditions at the terminal point of the control interval. How then is the new parameter information to be used in altering or updating the control policy? The brute force approach would be to extrapolate the parameter estimates forward to the end of the control interval and recompute the entire control sequence back to the present time. As might be imagined, the computing effort would be formidable. Moreover, as the extrapolations went further and further from the present the extrapolated parameters might be far from nominal. In this case it is not clear what significance the words "optimal feedback gains" would have.

Another possibility which was considered was an attempt to derive "parameter sensitivity coefficients" for the nominal feedback gains as a function of the parameter changes. Using these, the new gains could be computed as perturbations on the nominal gains based upon the parameter perturbations about the nominal. For single input, single output there are N gain coefficients while there are at most 2N parameters involved. Thus it would appear that the sensitivities could be found as an N x 2N matrix of coefficients which would map the coefficient differences δa into gain differences δK . But the gains at time k are not only a function of the present values of the parameters but also their future values since they are the solution of a difference equation starting at the terminal time. Thus it is once again necessary to have the extrapolated values of the parameters available. Moreover, the change in the feedback gains at time k is a function of all the future off nominal parameters and therefore, except for the case of constant coefficients, a sensitivity term for each future parameter value must be included. Again the computational problem is simply out of the question.

As a result of these considerations a compromise solution was reached. It was decided that instead of precomputing and storing the nominal gain sequence based upon the nominal parameter estimates, the nominal Riccati matrix and disturbance vector would be computed and stored according to the equations developed in Section 2.1. Then, at some time k, when new parameter estimates are obtained the gains for the next step would be computed by using the nominal Riccati matrix one time point in the future and the estimated parameters. That is, the control at time k to make the transition to time k + 1 would be computed via

$$m(k) = - \hat{K}(k)\, \hat{x}(k) - \hat{h}(k) \qquad (3.5\text{-}1)$$

$$\hat{h}(k) = \{Q(k) + \hat{B}^T \left[\overline{F}(k+1) + S(k+1)\right] \hat{B}(k)\}^{-1}$$

$$B^T \left[\overline{F}(k+1) + S(k+1)\right] d(k) + \overline{\zeta}(k+1) \qquad (3.5\text{-}2)$$

$$\hat{K}(k) = Q + \hat{B}^T(k) \left[\overline{F}(k+1) + S(k+1)\right] \hat{B}(k)^{-1}$$

$$\hat{B}^T(k) \left[S(k+1) + \overline{F}(k+1)\right] \hat{A}(k) \qquad (3.5\text{-}3)$$

The notation $\overline{F}(k+1)$, $\overline{\zeta}(k+1)$ indicates that these are the nominal Riccati Matrix, and disturbance vector. Notice that all other quantities used in equations (3.5-2) and (3.5-3), with the exception of the weighting matrices Q and S, are denoted as estimates made at time k. This specifically includes the deterministic disturbance, $\hat{d}(k)$, which can be estimated exactly like any other parameter. Indeed in the example which follows this is done.

What is the significance of this proposed computational scheme? In essence these are the equations which would result if all parameters were found to return to their nominal value at the next time interval and then remain there for the duration of the control interval. Although there is no reason that they should be expected to do this, the computational simplicity is attractive. As a result this method was chosen to easily incorporate the new estimates into the control.

The proposed system is now complete. It is seen to comprise two parts. The estimator and the controller. The estimator observes the inputs and outputs to the system and updates its estimate of both the system state and the system parameters. Based upon the altered parameters the new feedback gains are obtained in a one step computation using the precomputed nominal Riccati Matrix. Having obtained the gains, the control for the next step is computed as the matrix product of the gains and the states as expressed in (3.5-1). This completes the system analysis.

This system has been called "adaptive" by the author based upon this last ability to alter the control input based essentially upon new performance information.

(Since the estimator makes a new estimate based upon new observations of the system). In the remainder of this work the proposed method will be applied to a problem of engineering significance and some difficulty. Based upon the experience gained in the solution of this problem remarks about its general utility, usefulness, etc. can be made.

SECTION IV

CONTROL OF A FLEXIBLE LAUNCH VEHICLE

As an example of a problem of current interest, the control of a large flexible launch vehicle, of the Saturn type, was chosen. This section describes the pertinent characteristics of the vehicle as abstracted from Reference 1.

In overview, the problem may be stated as that of controlling the attitude of a large launch vehicle during flight. Control is exerted by gimballing four of the eight engines to provide torque. The actual vehicle pitch and pitch rate are assumed as the measured quantities. Because the vehicle is flexible the body cannot be considered rigid, rather it behaves as a vibrating free-free beam with a controllable torque applied at one end. As a result, the pitch and pitch rate sensors measure the local angles, not the fictitious rigid body angles. Moreover, gimballing of the engines to control pitch, etc. also excites the bending modes. The vehicle is subjected to wind gusts and the vehicle parameters change appreciably during the course of the flight. Finally, many of these parameters are only approximately known <u>a priori</u> since full scale testing is difficult. The control problem amounts to synthesizing a controller which can operate on the measurable signals, pitch and pitch rate, and control the vehicle attitude in spite of parameter variations and with aerodynamic disturbances present.

In order to restrict the problem to manageable size, while retaining a meaningful plant description the following general assumptions were made about the vehicle response:

1. Only first order bending mode effects are included.

2. No sloshing effects are included.

3. The engine gimbal angle is taken as the control input.

4. Viscous cross flow effects, occurring at high angles of attack, are ignored.

4.1 Vehicle Equations

Figure 4-1 defines the coordinates used for the rigid body. The pertinent rigid body equations are

$$\ddot{\phi}_R = - C_1 \alpha - C_2 \beta \tag{4.1-1}$$

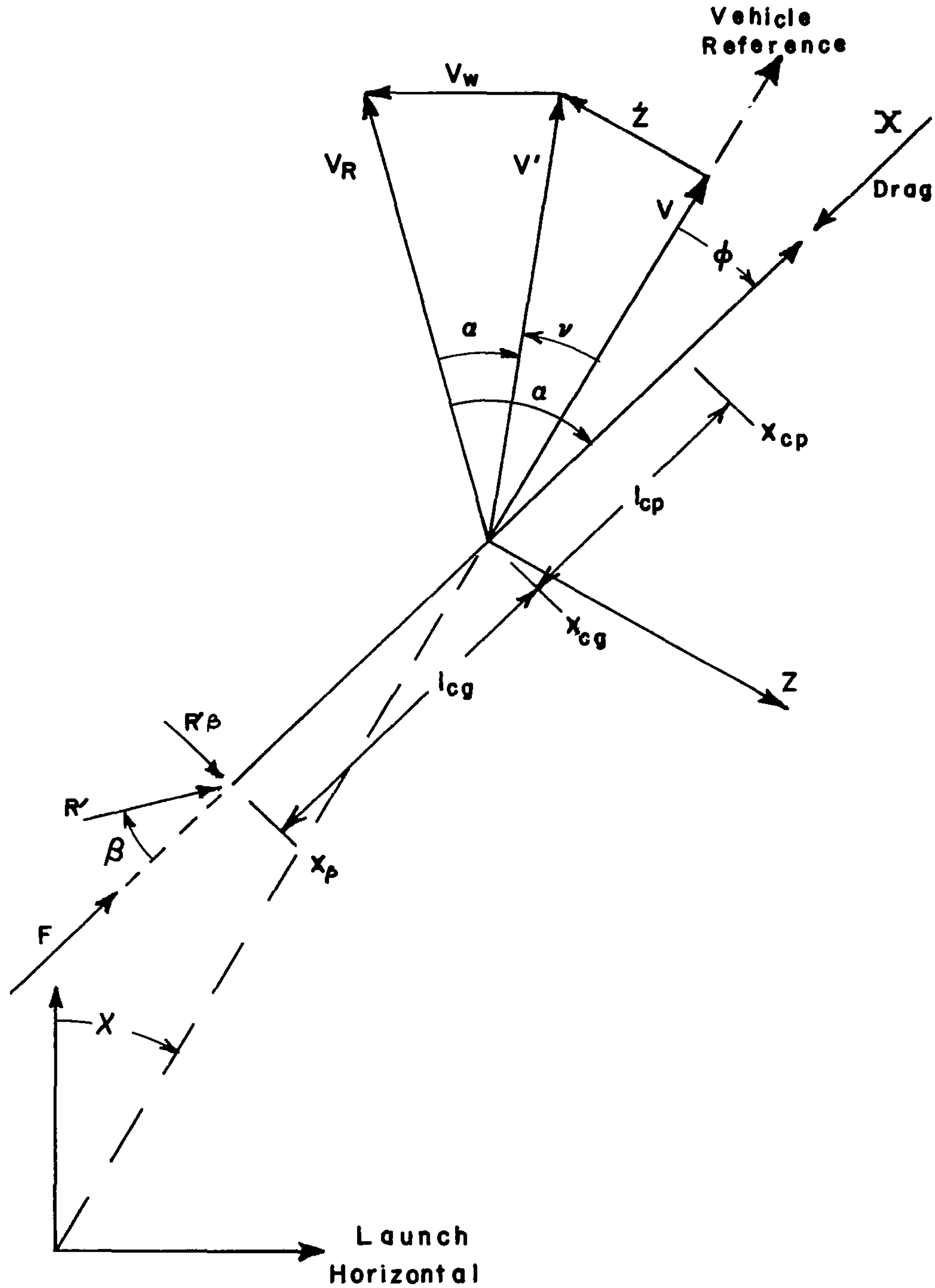

Figure 4.1 Coordinate Definitions

$$\ddot{z} = \frac{F - X}{M} \qquad (4.1\text{-}2)$$

$$\alpha = \phi_R - \dot{z}/V + \alpha_w \qquad (4.1\text{-}3)$$

where ϕ_R = Rigid body pitch angle.

$F - X$ = Accelerating Force.

$\dot{z}$ = Velocity normal to the reference.

α_w = Angle of attack induced by the wind.

C_1 = The aerodynamic torque coefficient.

C_2 = Engine Torque coefficient.

V = Nominal velocity

These two quantities may be written

$$C_1 = \frac{N'}{I_{xx}} \ell_{cp} = \frac{N'}{I_{xx}} (x_{cg} - x_{cp}) \qquad (4.1\text{-}4)$$

$$C_2 = \frac{R'}{I_{xx}} \ell_{cg} = \frac{1}{2} \frac{F}{I_{xx}} (x_{cg} - x_\beta) \qquad (4.1\text{-}5)$$

Equations (4.1-1) through (4.1-5) may be combined to obtain the differential equations for pitch and angle of attack.

$$\frac{d(\alpha - \alpha_w)}{dt} = - \frac{F - X}{MV} \phi_R + \dot{\phi}_R - \frac{N'}{MV} (\alpha - \alpha_w) - \frac{R'}{MV} \beta - \frac{N'}{MV} \alpha_w \qquad (4.1\text{-}6)$$

$$\frac{d\dot{\phi}_R}{dt} = - \frac{N' \ell_{cp}}{I_{xx}} \alpha - \frac{R' \ell_{cg}}{I_{xx}} \beta \qquad (4.1\text{-}7)$$

These equations describe the rigid body performance of the vehicle. The bending effects are included by considering the bending to be a vibration in "normal coordinates".[13] The angular deformation at any given point on the beam is then the amplitude of the normal coordinate multiplied by the mode shape coefficient for that position.

The equation of the first normal mode is that of a linear oscillator

$$\ddot{\eta}_1 + 2\zeta_1 \omega_1 \dot{\eta}_1 + \omega_1^2 \eta_1 = \frac{R'Y(x_\beta)}{M_1} \beta \qquad (4.1\text{-}8)$$

where η_1 = Normal mode amplitude

ω_1 = Mode frequency

ζ_1 = Mode damping

The right hand side of (4.1-8) illustrates how the mode is excited by the engine. The term $Y(x_\beta)$ is the mode shape at the gimbal station and M_1 is the "generalized mass" for the first bending mode. The sensors measure the angular position and rate locally, that is at their location. This angle is the sum of the rigid body angle and the bending angle. Hence the sensors measure

$$\phi_s = \phi_R + \phi_B = \phi_R - Y_1'(x_\phi)\, \eta_1 \qquad (4.1\text{-}9)$$

$$\dot{\phi}_s = \dot{\phi}_R + \dot{\phi}_B = \dot{\phi}_R - Y_1'(x_{\dot{\phi}})\, \dot{\eta}_1 \qquad (4.1\text{-}10)$$

Where x_ϕ and $x_{\dot\phi}$ are the positions of the pitch and pitch rate sensors respectively on the vehicle.

Equations (4.1-6) through (4.1-10) may be combined to obtain a matrix description of the vehicle which is required to apply the theory developed previously. Generally

$$\dot{x} = A\,x + B\,m + \Gamma u$$

$$y = C\,x \tag{4.1-11}$$

where the matrices A, B, C and Γ are written out as

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\dfrac{N'\ell_{cp}}{I_{xx}} & 0 & 0 \\ -\dfrac{F-X}{MV} & 1 & -\dfrac{N'}{MV} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\omega_1^{\;2} & -2\zeta_1\omega_1 \end{bmatrix} \tag{4.1-12}$$

$$B^T = \begin{bmatrix} 0 & -\dfrac{R'\ell_{cg}}{I_{xx}} & -\dfrac{R'}{MV} & 0 & \dfrac{R'Y_1(x_\beta)}{M_1} \end{bmatrix} \tag{4.1-13}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & -Y_1{}'(x_\phi) & 0 \\ 0 & 1 & 0 & 0 & -Y_1{}'(x_{\dot\phi}) \end{bmatrix} \tag{4.1-14}$$

$$\Gamma^T = \begin{bmatrix} 0 & -\frac{N' \ell_{cp}}{I_{xx}} & -\frac{N'}{MV} & 0 & 0 \end{bmatrix} \qquad (4.1\text{-}15)$$

$$x^T = \begin{bmatrix} \phi & \dot{\phi} & (\alpha - \alpha_w) & \eta_1 & \dot{\eta}_1 \end{bmatrix} \qquad (4.1\text{-}16)$$

A block diagram of these equations is shown in Figure 4-2. The α_w term is seen to play the role of disturbance. The entire question of wind disturbance is discussed in Section 4.3. Because the analysis has been carried out in terms of discrete plants it was necessary to convert the continuous description given by (4.1-12) through (4.1-16) to the discrete case. The manner in which this was accomplished is discussed briefly in 4.2.

4.2 Discrete Representation

It is desired to convert the vehicle differential equations to their equivalent difference equations. Specifically, a set of matrices A^*, B^*, C^*, Γ^* is sought such that

$$x(k + 1) = A^* x(k) + B^* m(k) + \Gamma^* u(k)$$

$$y(k) = C^* x(k) \qquad (4.2\text{-}1)$$

gives the same response at the sampling intervals as does the system of (4.1-11), under the assumption that m and u are constant over the interval (k, k + 1). Since there are five

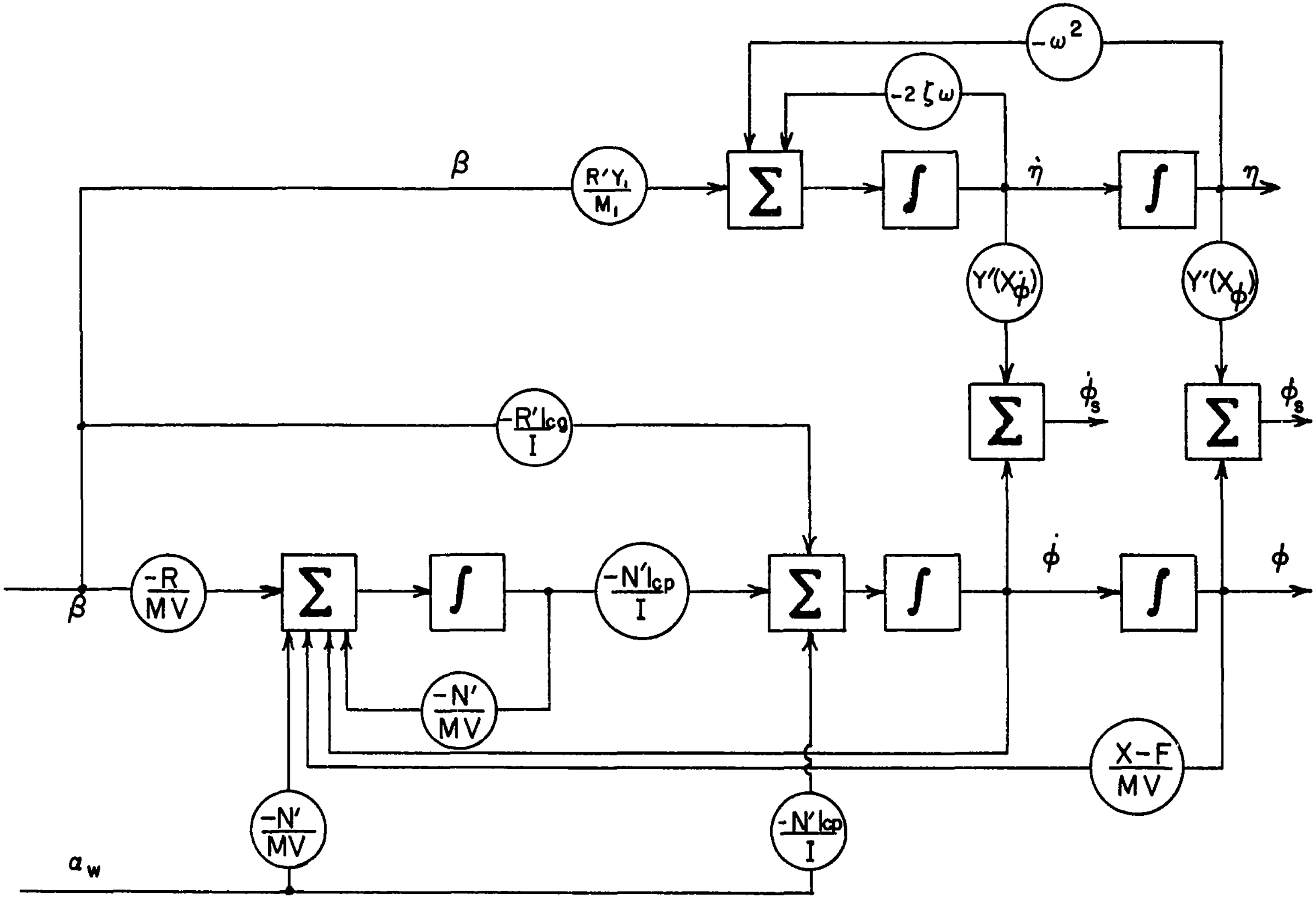

Figure 4.2 System Block Diagram

state variables, analytic conversion either through use of z transforms or of fundamental matrix methods was clearly out of the question. Rather, a computer program was written to convert the equations by successive integrations.

First, it is readily apparent that C^* is equivalent to C, since these relations are algebraic. To find A^* note that if m and u are identically zero and the vector x(0) is zero everywhere except for a 1 in the jth row then the value of x, from equation (4.1-11) one sample period (T seconds) later is

$$x(T) = a_j \tag{4.2-2}$$

where a_j = jth column of A^*. By varying j from 1 to n the complete matrix A^* is thus found. Similarly by setting $x(0) \equiv 0$, $u \equiv 0$, $m(t) = 1$ and integrating (4.1-11) over a sample period B^* is obtained as the value of x(T). Finally Γ^* is found from setting $x(0) = 0$, $m(t) = 0$, $u(t) = 1$ and integrating.

The data for the continuous case was obtained at eight second intervals from Reference 1. Some of the important curves are plotted in Figures 4-3 to 4-7. The data was then hand converted to the matrix formulation given by equations (4.1-12) to (4.1-16). Sensor locations of

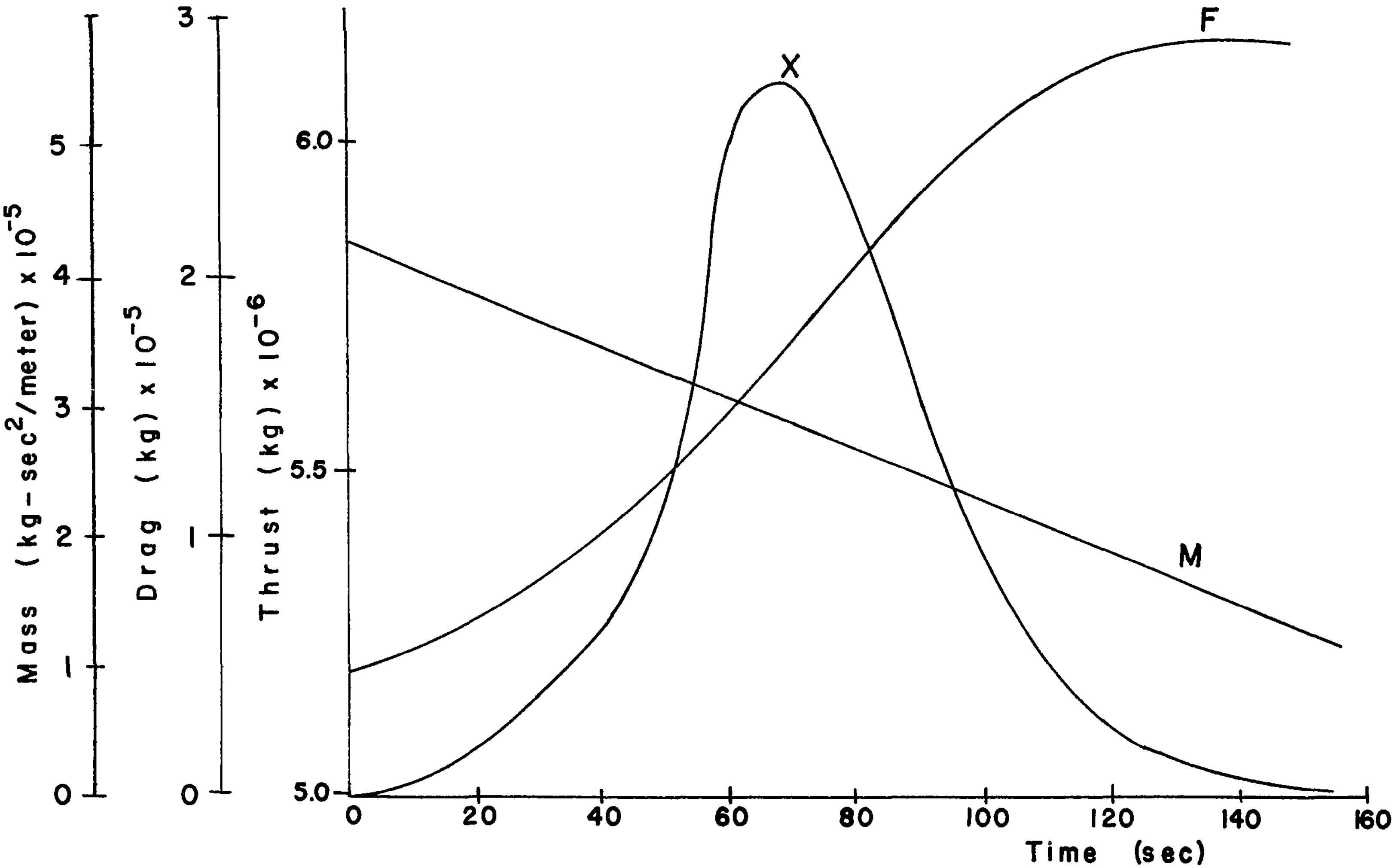

Figure 4.3 Nominal Vehicle Mass, Drag and Thrust Profiles versus Time

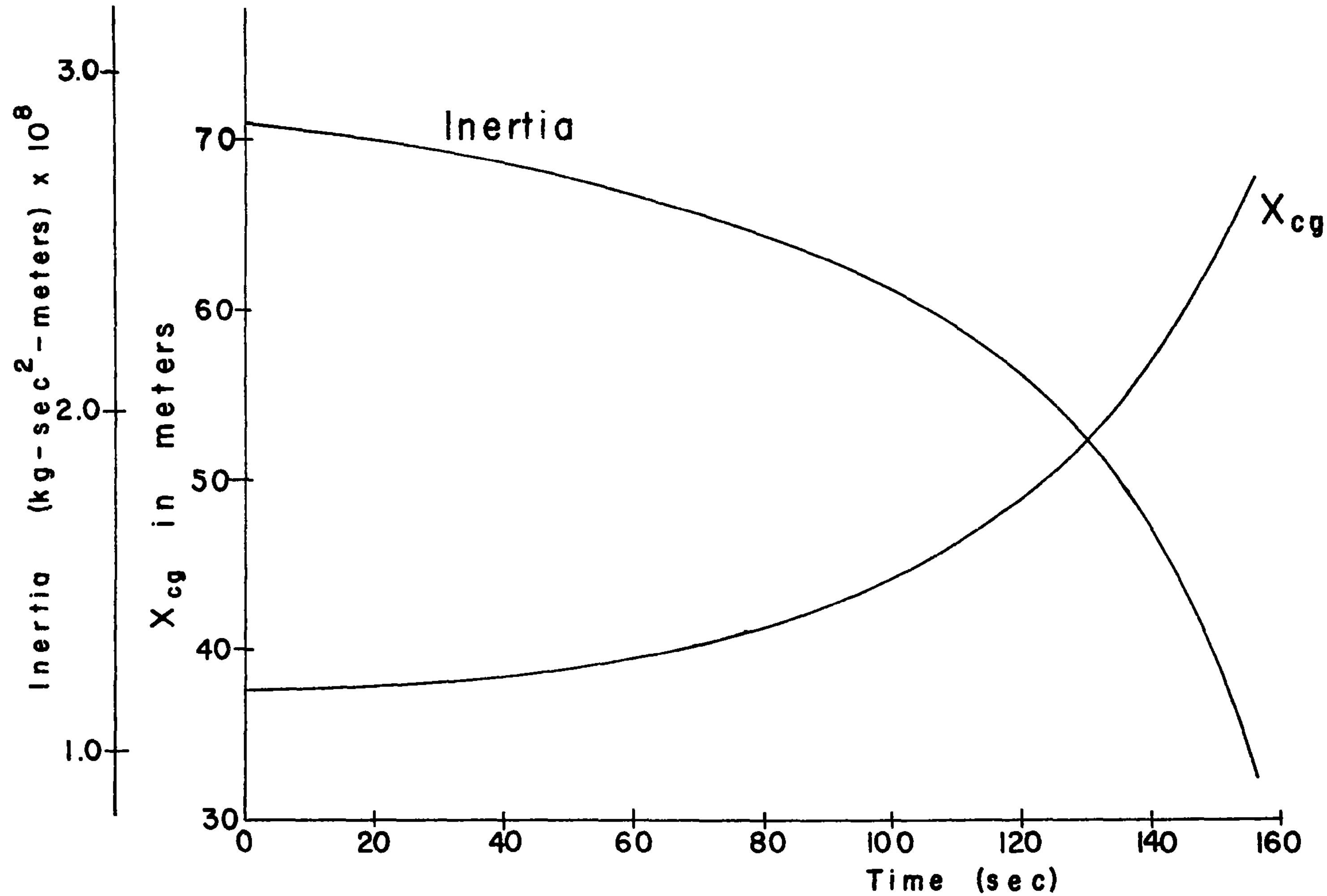

Figure 4.4 Nominal Pitch Inertia and C. G. Location versus Time of Flight

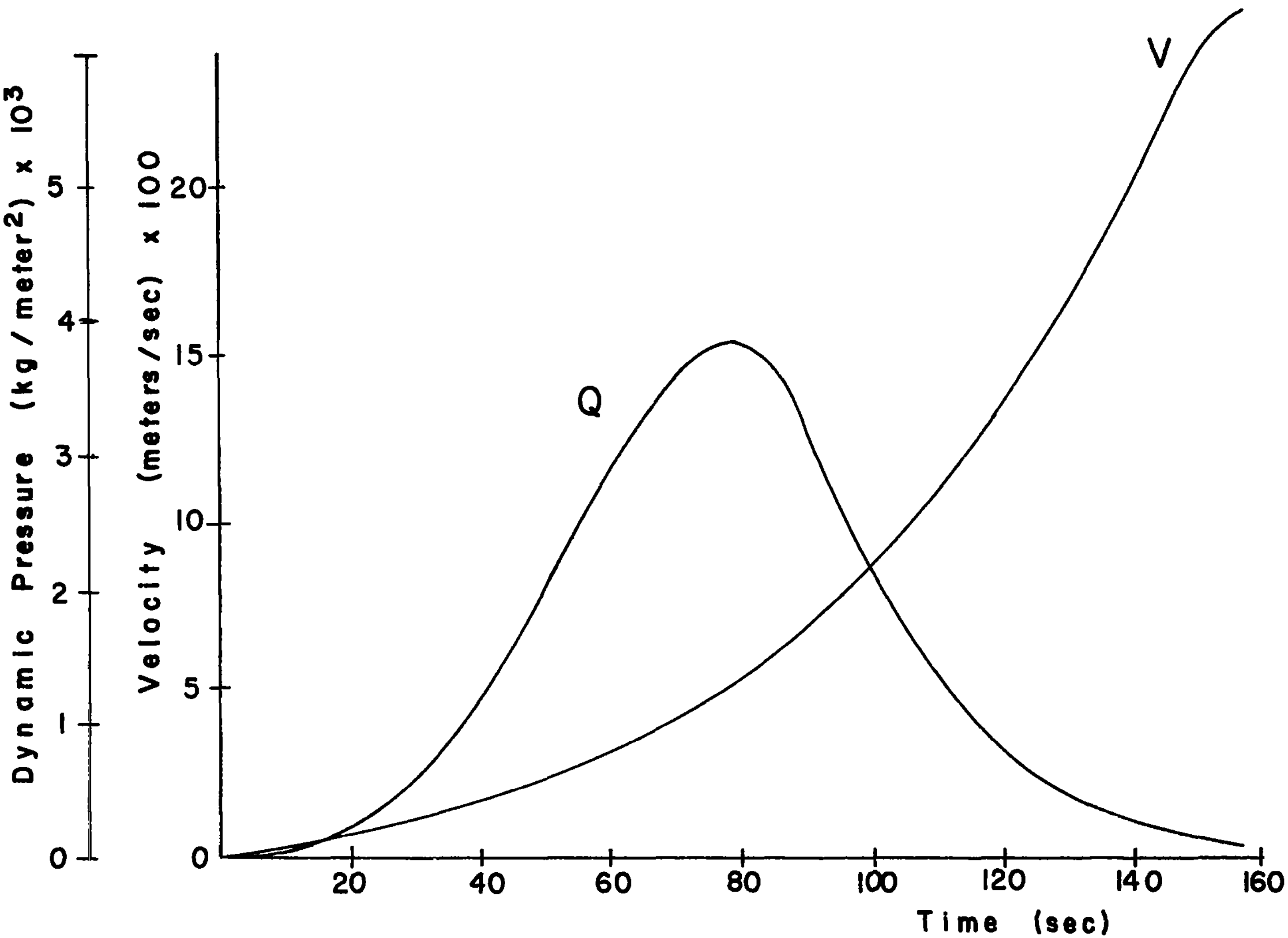

Figure 4.5 Dynamic Pressure and Velocity for Nominal Flight versus Time

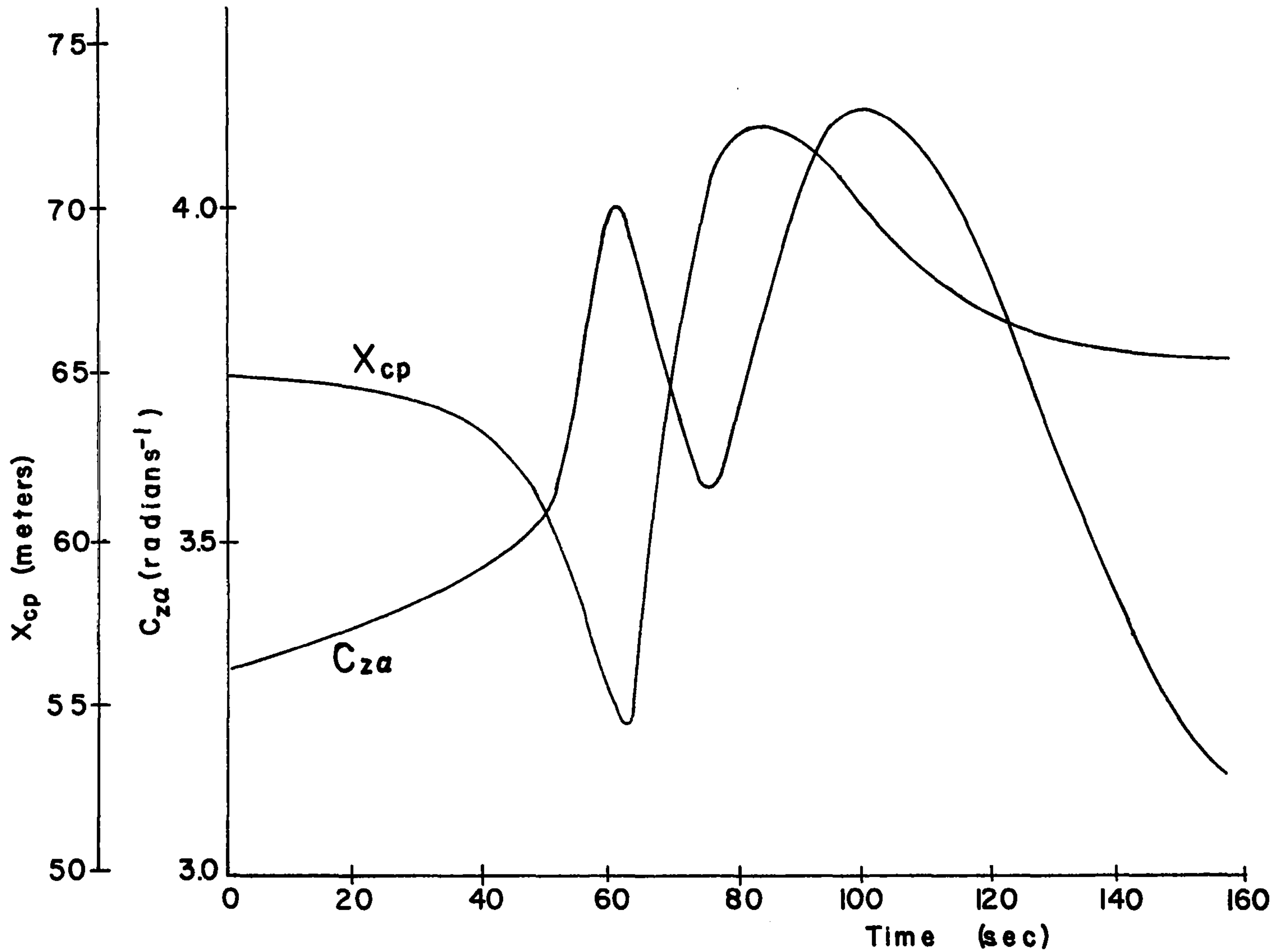

Figure 4.6 Center of Pressure and Normal Force Coefficient

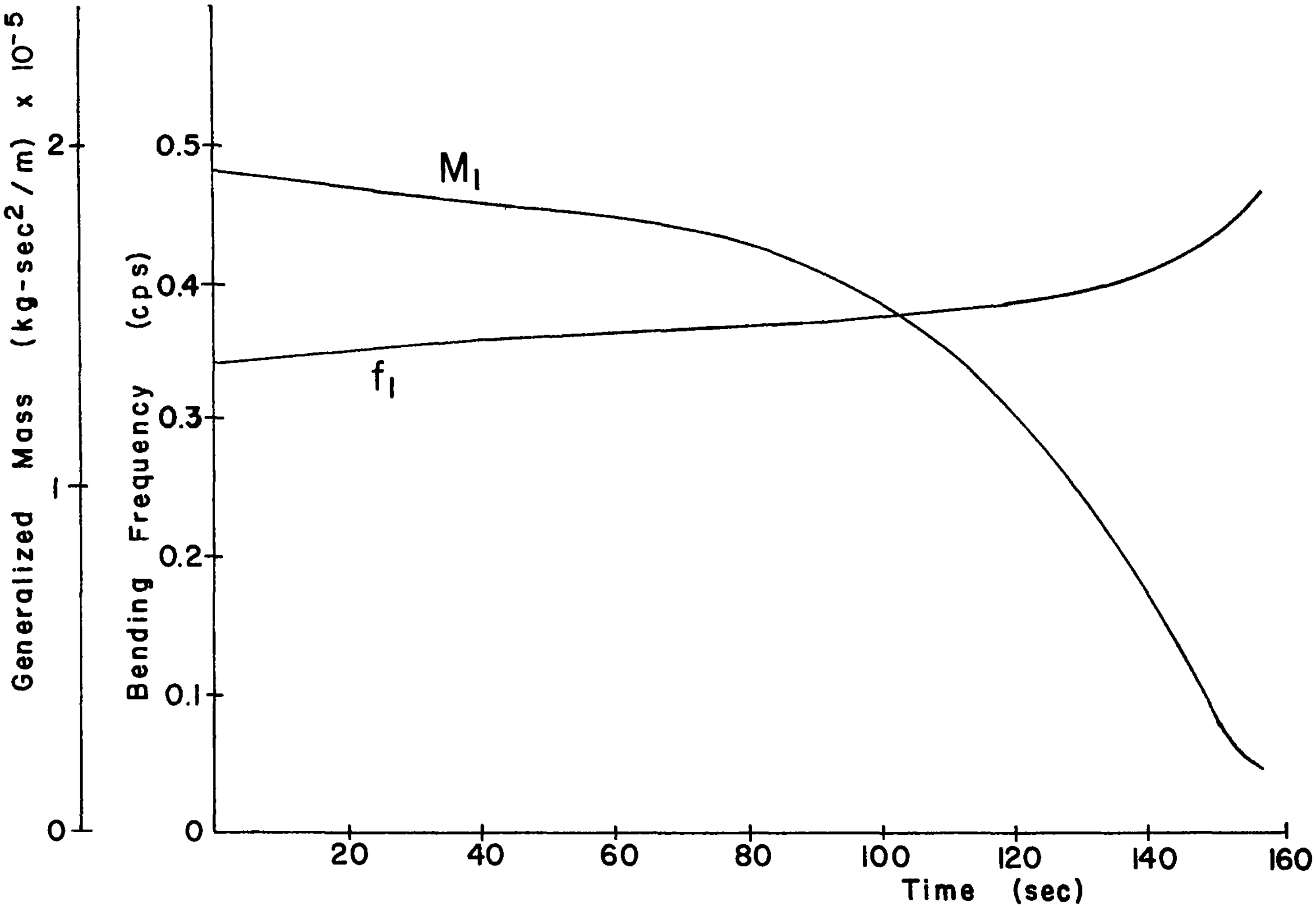

Figure 4.7 Generalized Mass and Natural Frequency for First Bending Mode

$$x_{\phi} = 120 \text{ meters} \tag{4.2-3}$$

$$x_{\dot{\phi}} = 50 \text{ meters} \tag{4.2-4}$$

were chosen from a chart of allowed sensor locations. These locations are actually not optimal from the standpoint of mode shape nulls but rather were chosen to illustrate the systems ability to filter out the bending frequencies. Other constants and miscellaneous relations used were:

$$\begin{aligned} x &= 2.54 \text{ m} \\ A &= 79 \text{ m}^2 \\ N' &= C_{z\alpha} q A \end{aligned} \tag{4.2-5}$$

The A, B, C and Γ data was then punched on cards. The conversion program reads the cards, sets the initial conditions as described above and employs a Runge-Kutta[24] integration routine to successively obtain x(T). Since it was decided to use

$$\Delta T = 1 \text{ sec} \tag{4.2-6}$$

in the sampled representation the program then performed linear interpolation of the A^*, B^*, C^* and Γ^* matrices, to get their values at each second, rather than their values each eight seconds. Finally, the results were punched out to form the booster master description.

4.3 Wind Disturbances

The vehicle description given in Reference 1 also contains a discussion of wind data. The following is a summary of that material as it was used in this study. Figure 4-1 shows the geometry of the vehicle inertial velocity v, the wind vector w and the resultant angle of attack, α_w , due to the wind. From this figure it is clear that

$$\alpha_w = \frac{w \cos x}{v - w \sin x} \qquad (4.3\text{-}1)$$

where x is the tilt angle measured from vertical at launch. Notice that the wind is assumed to be normal to the launch vertical. Although Reference 1 contains extensive discussion of wind shears and embedded jets it was decided that for this study α_w would be treated as having two components, such that

$$\alpha_w = \alpha_{wss} + \alpha_{wr} \qquad (4.3\text{-}2)$$

The component α_{wss} would represent the steady state wind component, while α_{wr} would be a random component which would represent wind gusts. Further, it was assumed that the wind gusts were uncorrelated with respect to the 1 sec sampling intervals used.

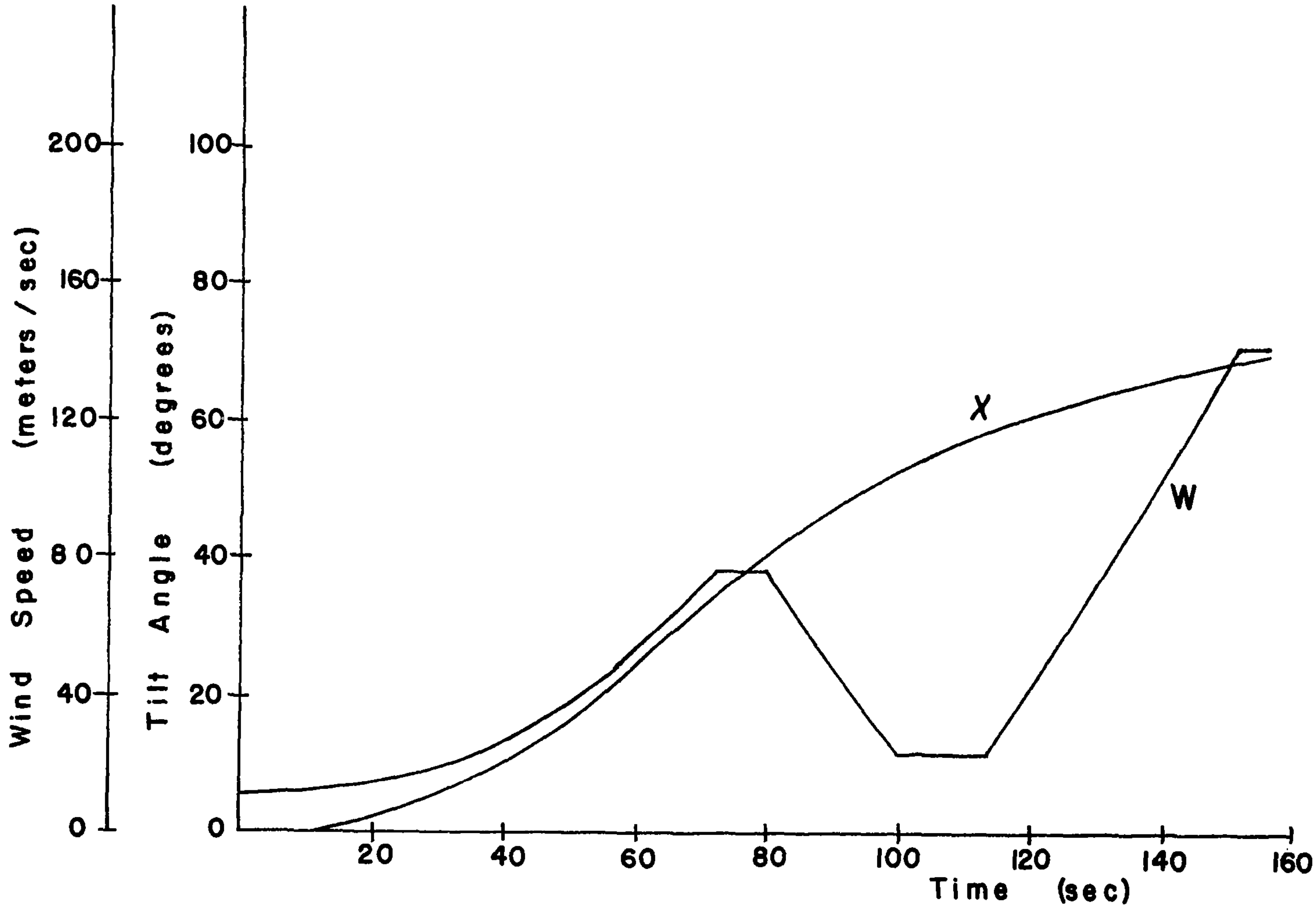

Figure 4.8 Wind Speed and Tilt Angle vs Time

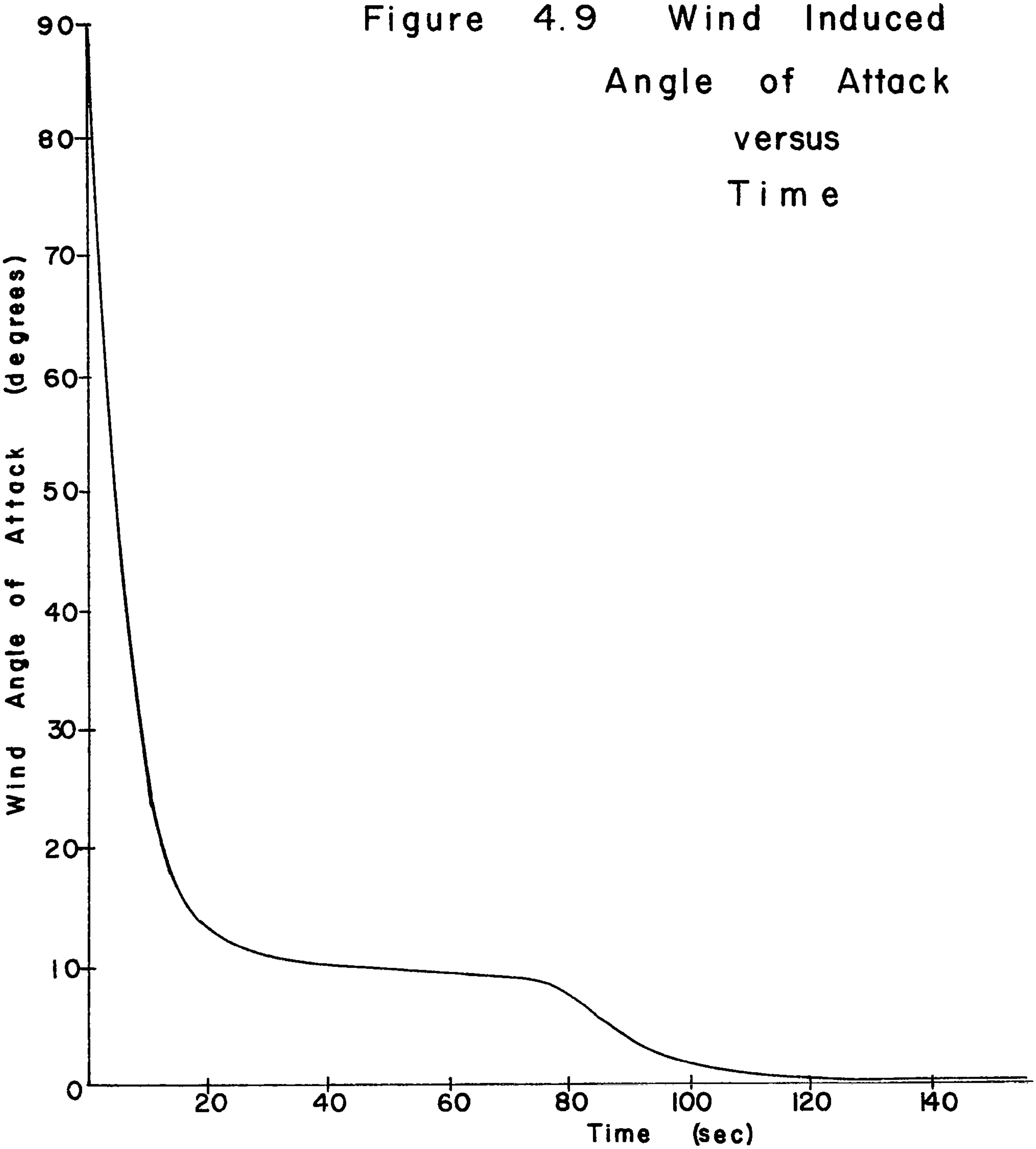

Figure 4.9 Wind Induced Angle of Attack versus Time

The steady state components were computed using equation (4.3-1) and the data on v and x supplied for Model Vehicle No. 2. These are shown plotted in Figures 4-3 and 4-5. For w, the wind speed, a wind speed profile envelope was chosen. This profile is reproduced as Figure 4-8. The profile used is a worst case profile in the sense that it represents the envelope of 95% probability of occurrence of wind speed over the entire year. The resulting worst case α_w is shown, plotted in Figure 4-9. This profile was included as part of the vehicle master description. To get other cases, the worst case profile could be multiplied by a suitable scale factor.

4.4 Evaluation of the Vehicle Control System

In order to evaluate the performance of the controller developed in the previous sections for the flexible launch vehicle problem the entire system was simulated on a digital computer. Actually due to the complexity of the problem three simulations were used. The first one had no parameter identification or feedforward term but consisted solely of the classic Kalman filter plus feedback gain matrix configuration. This simulation was used because it was felt that a reasonable choice of the performance index weighting matrices Q and S could be made more easily if the effects of identification were removed. This system would also provide a benchmark for comparison with the adaptive system performance. The second simulation, actually a minor modification of the first, incorporated the feedforward term. Finally, the third simulation incorporated the adaptive system. Appendix C contains a flow chart for each, together with some descriptive material.

Numerous runs were made with the first system, since it was the simplest, and considerable insight into the behavior of a Kalman filter-gain matrix controller was obtained. The usual procedure was to utilize nominal vehicle data and exact matching of assumed and actual noise variances while adjusting the weighting matrices Q and S. When apparently satisfactory performance was obtained the resultant control was evaluated for off nominal vehicle parameters and discrepancies between the design and actual noise levels for both the wind and the sensors. The initial conditions chosen for all cases were

$$\begin{aligned} \phi &= 5^{\circ} & \eta &= .02 \text{ meters} \\ \dot{\phi} &= 0 & \dot{\eta} &= 0 \\ \alpha &= \alpha_w \end{aligned} \qquad (4.4\text{-}1)$$

Most systems were designed assuming the noise variance of the wind to be 0.5° and a sensor disturbance matrix of

$$L = \begin{bmatrix} .25 & 0 \\ 0 & .25 \end{bmatrix} \tag{4.4-2}$$

Some of the early runs were made with twice these noise levels but it was felt that such levels might be unrealistic. Appendix D contains a log of all simulations, together with brief remarks.

Preliminary runs indicated that a choice of

$$Q = 0.1 \qquad S = \begin{bmatrix} 2000 & & & & \\ & 1000 & & & \\ & & 0 & & \\ & & & 50 & \\ & & & & 25 \end{bmatrix} \tag{4.4-3}$$

gave reasonably good performance. The transient response to the initial error was rapid, requiring about 10 seconds, but the steady wind caused an error of about 1° in pitch in the neighborhood of max q (80 seconds). In an effort to improve this the weighting on the bending terms (x_4 and x_5) was reduced. This resulted in very high vibration in the bending mode. Even more interesting was the discovery that a reduction of Q to .01 resulted in a virtually identical (to two decimal places) performance to that obtained when the weighting of (4.4-3) was used. This result, obtained as run I-7, indicated that the problem was not lack of control effort, at least for small pitch angles, but a trade-off between control of the pitch angle and excitation of the bending. When the bending weight was dropped to 5 and 2.5 for position and rate respectively the vibration was severe and there was little improvement in the pitch angle. Another interesting facet of the problem appeared for the latter weighting when a run was made with the actual sensor noise reduced by a factor of 5 from the design value of (4.4-2). The vibration actually got worse (run I-4) contrary to expectation. This phenomena was observed several times (runs I-4, III-4, III-5) and appeared whenever very high pitch weightings were used. The heuristic argument explaining this is that the high weighting results in a very tight pitch loop. Any off nominal noise level causes errors in the Kalman filter and the resultant noise propagates through the bending mode and the tight loop.

As a result of the Block I runs it was decided that the bending weighting had to be some fraction of the rigid body weighting in the neighborhood of equation (4.4-3) to minimize vibration. Figure 4-10 shows the performance of the system for these weightings. The maximum dynamic pressure occurs at about 80 seconds. Notice the decided peak in the control effort curve in this region as the controller has to counteract the wind disturbance. Also notice that the bending vibration goes up in this area. The initial conditions are as given in (4.4-1) for these curves. After a sharp transient in the bending induced by the controller reducing the pitch error the bending mode has practically no oscillation. Due to the steady wind both the pitch angle and the bending exhibit an offset, which becomes more pronounced as peak dynamic pressure (max q) is approached. For clarity in the figure $-\beta$ is plotted to keep it separate from ϕ without having to make a separate plot.

The offset in pitch and bending was naturally considered undesirable. Since it had been established that raising the pitch weighting was not satisfactory the next runs (Block II) experimented with weighting angle of attack and adding a non-zero off diagonal term to the matrix S. Weighting angle of attack invariably produced a large degree of instability, apparently due to the disturbance input. The next attempt to decrease the offset was to include a term $k\ x_1\ x_4$ in the performance index by making $S_{14} = S_{41} \neq 0$.

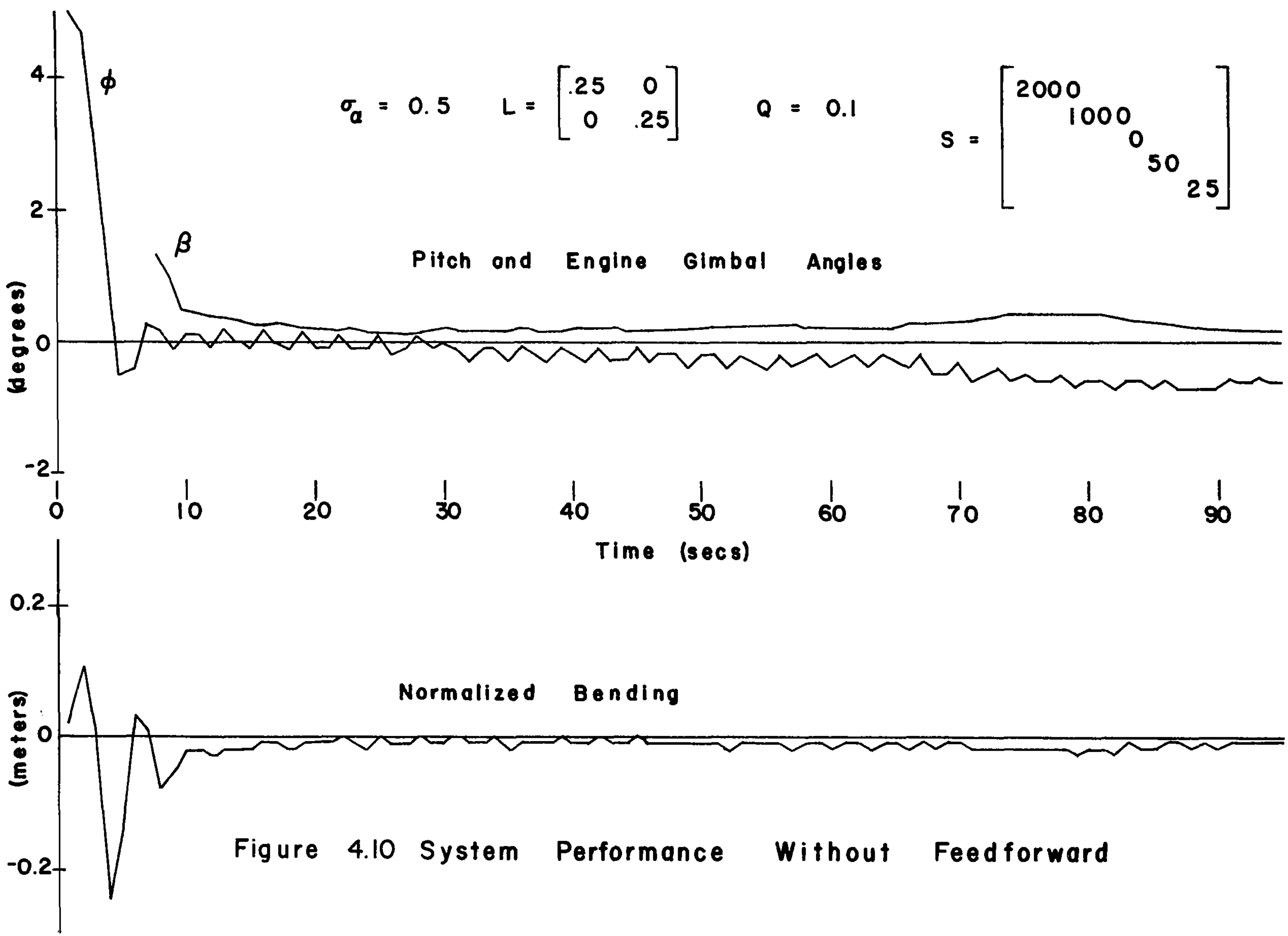

Figure 4.10 System Performance Without Feedforward

For small values, say 5 to 10 this did indeed give a slight improvement. When the term was increased however, the control became very oscillatory. Early success with this cross term led to some trials with an increase in the pitch weighting. These further reduced the pitch error, but as before were extremely sensitive to the noise levels actually present relative to those assumed in the system design (Block III).

Flights with off nominal vehicle data indicated that the control system with the weights of equation (4.4-3) was more than adequate to compensate for $\pm$ 20% changes in the aerodynamic lift coefficient. On the other hand, variations in bending frequency resulted in severe vibration in the bending modes (Block IV). This is probably due to the fact that the rigid body mode can be controlled and measured through a tight loop whereas the bending is not easily measured. As a result it was decided to concentrate upon the off nominal bending case. In particular the -20% bending frequency was considered because this moves the bending down further into the pass band of the controller. Once again increase in the pitch weighting did not improve the situation with respect to bending frequency although the off nominal aerodynamic cases showed no degradation.

With the addition of the cross product term it was found that reasonable noise insensitivity and pitch control could be obtained by setting

$$Q = 0.1 \qquad S = \begin{bmatrix} 10000 & 0 & 0 & 5 & 0 \\ 0 & 5000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 50 & 0 \\ 0 & 0 & 0 & 0 & 25 \end{bmatrix} \tag{4.4-4}$$

However, some vibration was induced when the noise levels shifted from their nominal values (Block 5). For this reason a compromise was chosen, and for simplicity the cross product term dropped. The final weighting factors chosen were

$$Q = 0.1 \qquad S = \begin{bmatrix} 4000 & & & & \\ & 4000 & & & \\ & & 0 & & \\ & & & 50 & \\ & & & & 25 \end{bmatrix} \tag{4.4-5}$$

Figure 4-11 shows the nominal system performance for this choice of weighting factors. There has been a slight improvement in the pitch error at the expense of the bending displacement. Figure 4-12 shows the same system for a

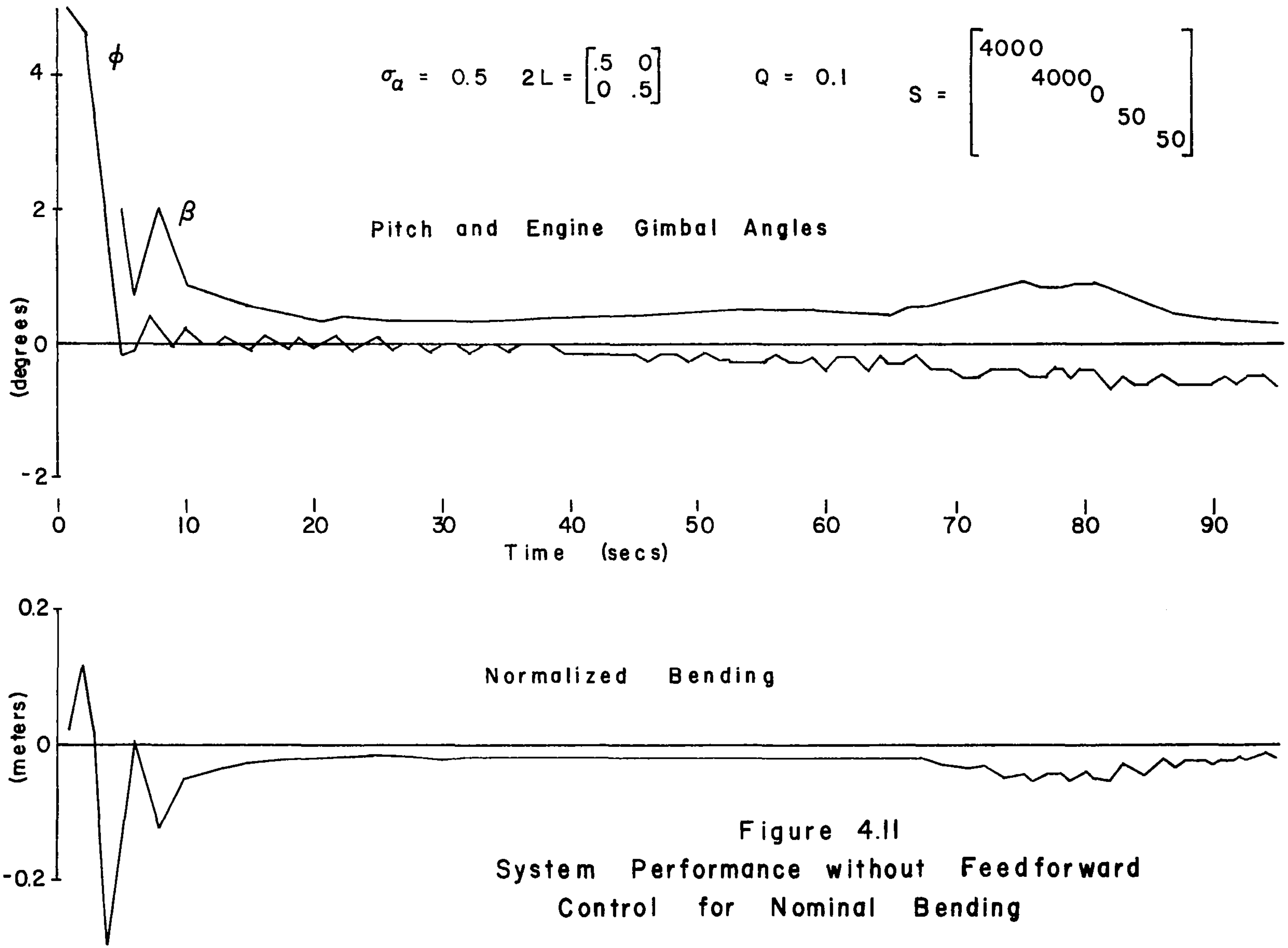

Figure 4.11
System Performance without Feedforward Control for Nominal Bending

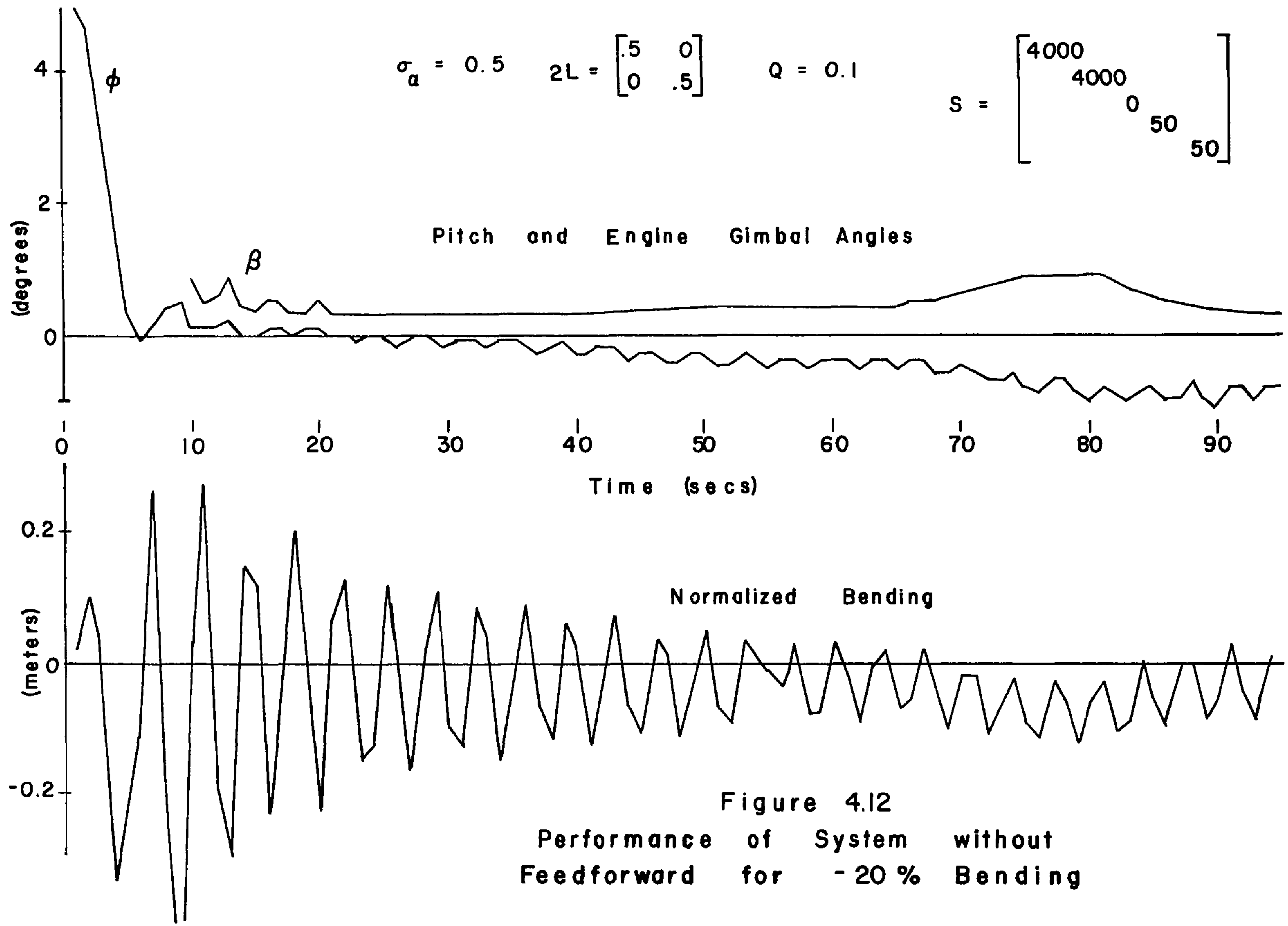

Figure 4.12
Performance of System without
Feedforward for -20 % Bending

20% reduction in bending frequency. Notice the high vibration.

Figures 4-13 and 4-14 show the system performance for off nominal wind gust noise and sensor noise respectively. No significant degradation in performance is observed. Indeed, for the reduced sensor noise the only noticeable change was an improvement in the estimator. Reduced wind noise causes a smoother flight to result.

Figure 4-15 shows the advantages of the feed-forward term in reducing the wind offset. The most easily noticed difference is in the gimbal angle. The peak in the control effort near maximum dynamic pressure is missing. This is because the feedforward term possesses prediction and can attempt to correct for the disturbance before it actually occurs. Both the bending and the pitch are reduced although the pitch reduction is small. On the other hand the bending deflection is cut in half in the region of maximum dynamic pressure. Figure 4-16 shows the problem with this controller for -20% bending frequency. Notice the high vibration, although pitch control seems only slightly degraded. This seems partly to be due to the characteristics of the estimator. This figure, unlike the others, also has the estimated bending plotted. The

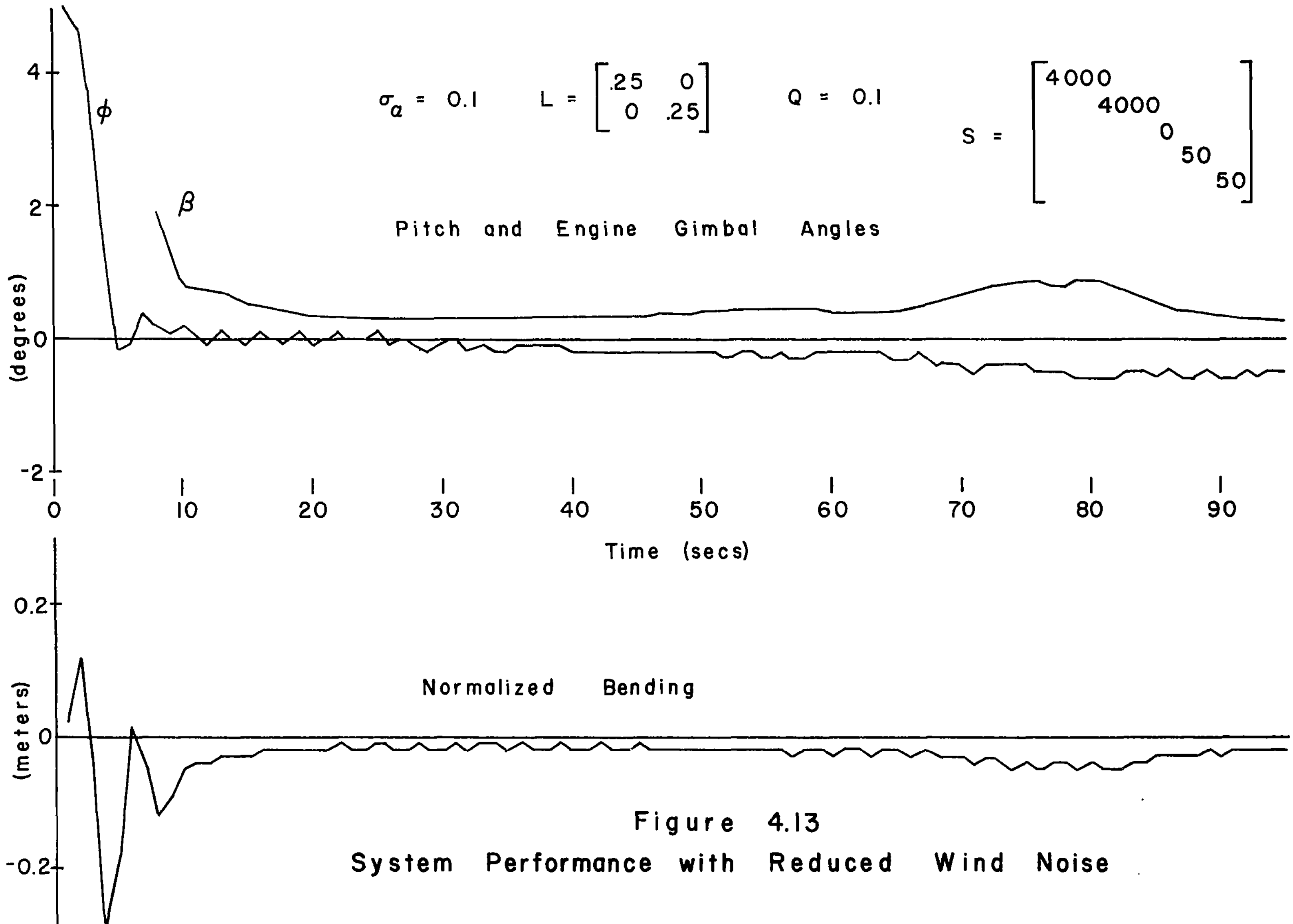

Figure 4.13
System Performance with Reduced Wind Noise

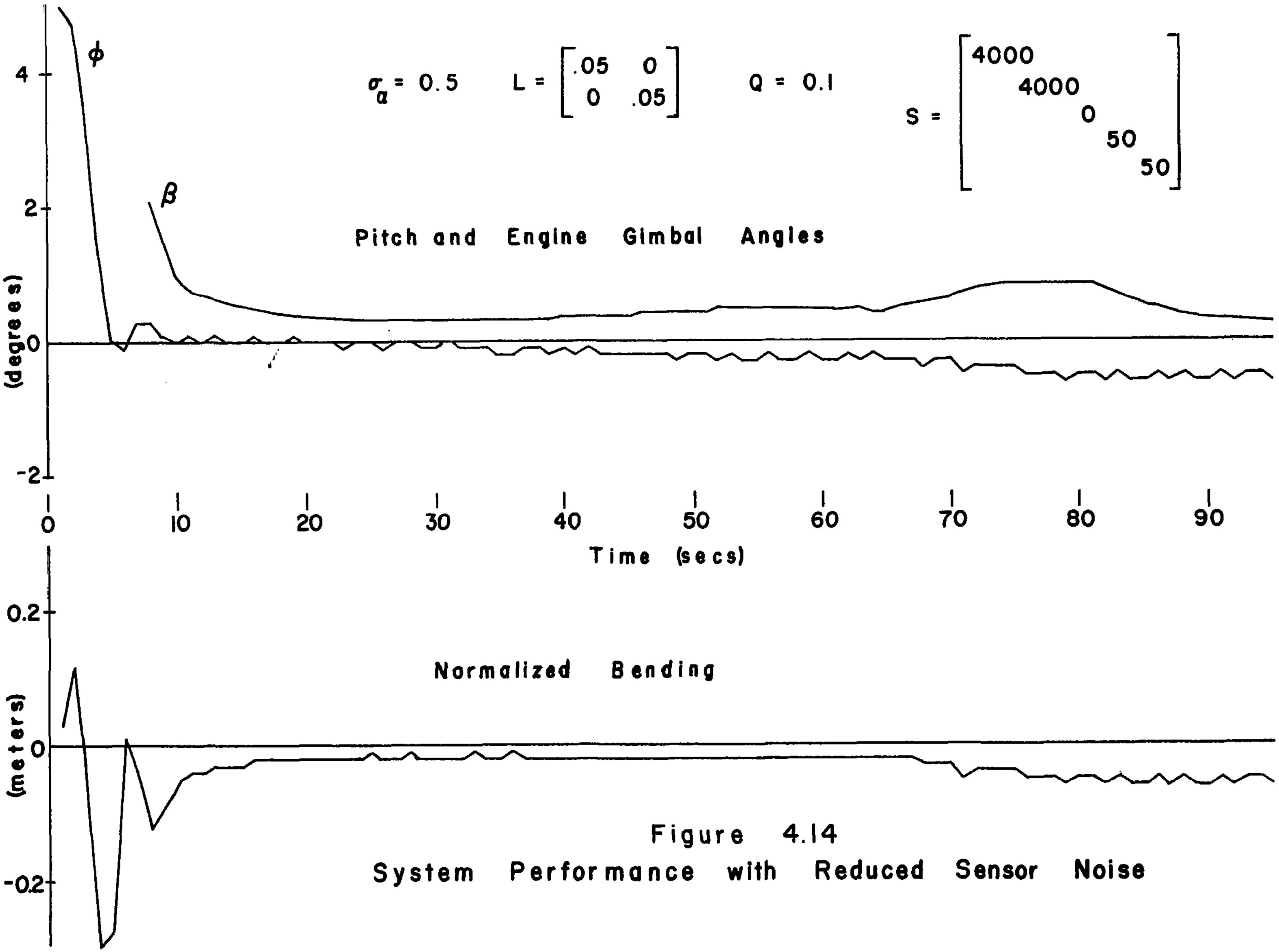

Figure 4.14
System Performance with Reduced Sensor Noise

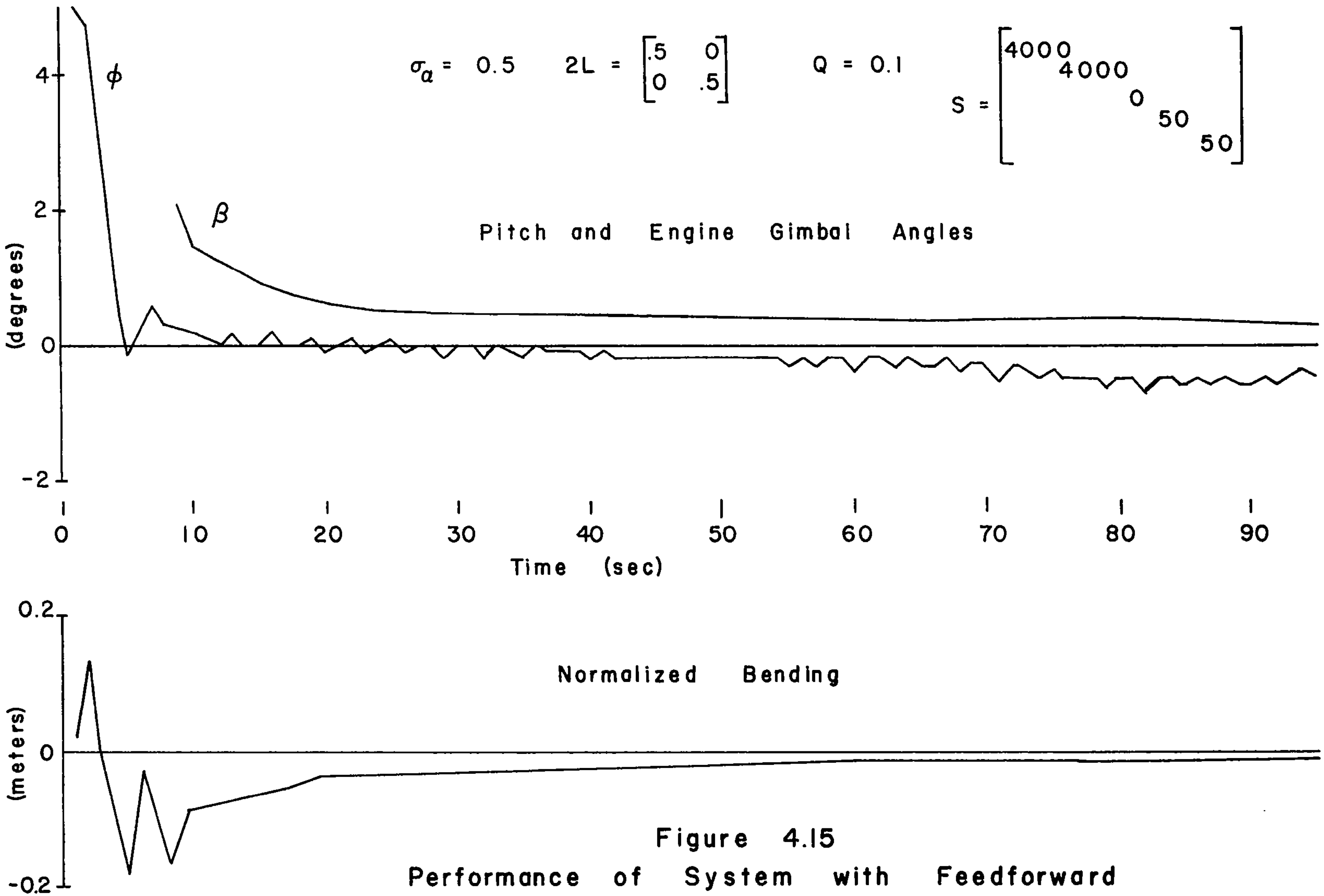

Figure 4.15
Performance of System with Feedforward Included for Nominal Bending

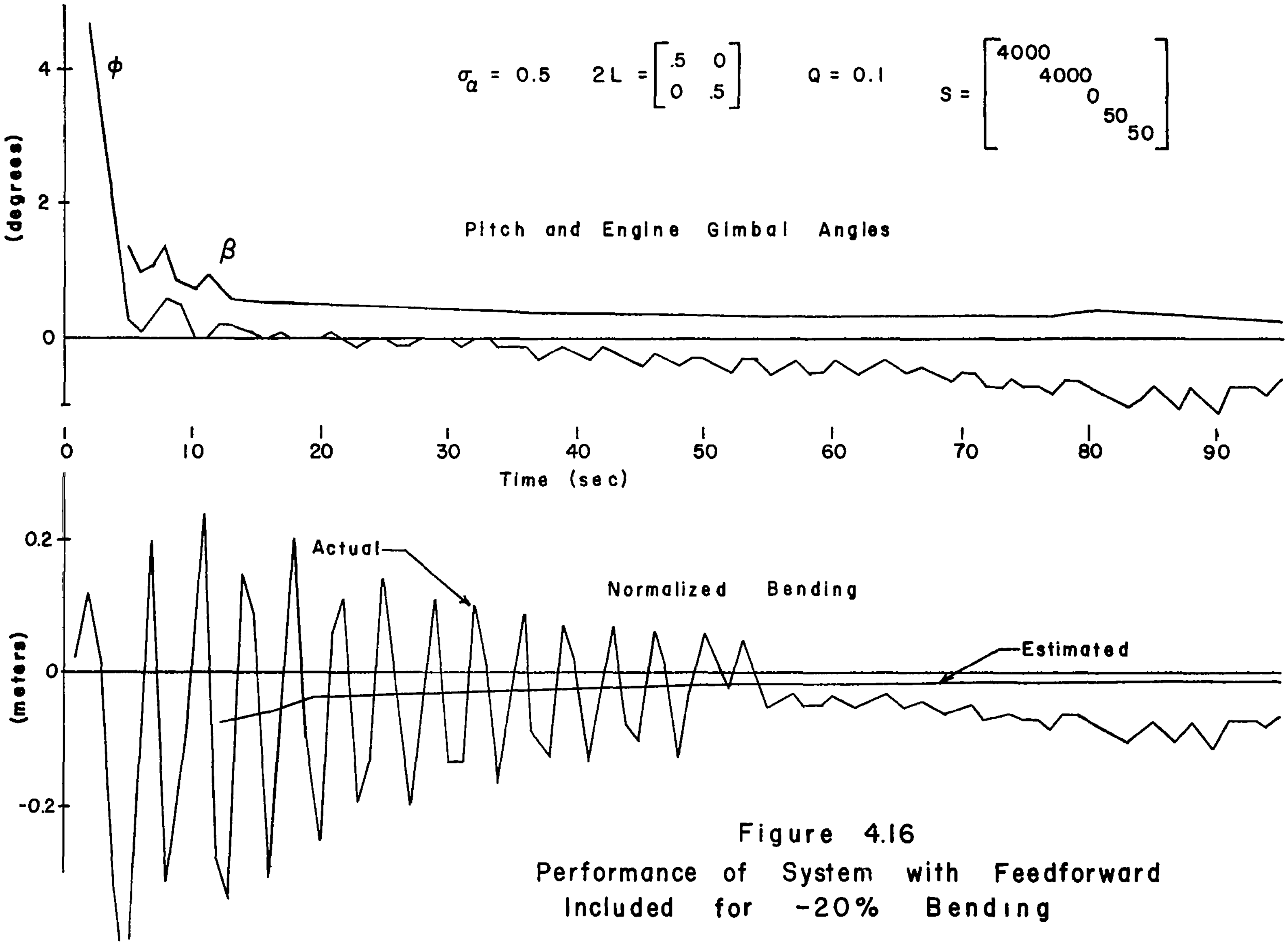

Figure 4.16
Performance of System with Feedforward
Included for -20% Bending

estimator output follows closely the curve the actual bending did for nominal data! (Compare to Figure 4-15.) This phenomena was also observed when the feedforward term was not present in the controller. The vibration amplitude seems to have an envelope with a time constant of about 40 seconds. In the area of maximum dynamic pressure the vibration damps out, or the frequency goes up and the amplitude comes down due to the aerodynamic forces.

These results clearly indicate the necessity for some sort of adaptive system which can account for the variations in the bending frequency. Because the aerodynamic variations had had little effect upon system performance the identification was limited to the discrete parameters associated with the bending and the unmeasurable angle of attack due to the wind. Although the continuous vehicle has only two parameters which are functions of the bending frequency, the discrete version has four parameters which are functions of the frequency. If the unknown mean of the wind disturbance α_w is added a total of 5 parameters to be identified are obtained. Define the state vector as

$$x^T = \left[\phi,\ \dot{\phi},\ \alpha - \alpha_w\ ,\ \eta,\ \dot{\eta}\right] \qquad (4.4\text{-}6)$$

Then the augmented state is

$$x_A^T = \left[\phi, \ \dot{\phi}, \ \alpha - \alpha_w, \ \eta, \ \dot{\eta}, \ a_{44}, \ a_{45}, \ a_{54}, \ a_{55}, \ \alpha_w \right] \quad (4.4\text{-}7)$$

where a_{44} etc. are the four discrete parameters related to the bending frequency. The differential input vector and transition matrix are given by equation (4.4-8) and (4.4-9) shown on the next page.

$$\left(\frac{\partial f}{\partial m} \right) = \left[\Gamma^T \quad \gamma_{a_{44}} \gamma_{a_{45}} \gamma_{a_{54}} \gamma_{a_{55}} \gamma_{\alpha_w} \right] \quad (4.4\text{-}8)$$

$$\left.\frac{\partial f}{\partial x}\right|_{x(k|k)} =$$

$$\begin{bmatrix}
a_{11} & a_{12} & a_{13} & 0 & 0 & 0 & 0 & 0 & 0 & d_1 \\
a_{21} & a_{22} & a_{23} & 0 & 0 & 0 & 0 & 0 & 0 & d_2 \\
a_{31} & a_{32} & a_{33} & 0 & 0 & 0 & 0 & 0 & 0 & d_3 \\
0 & 0 & 0 & a_{44}(k|k) & a_{45}(k|k) & x_4(k|k) & x_5(k|k) & 0 & 0 & 0 \\
0 & 0 & 0 & a_{54}(k|k) & a_{55}(k|k) & 0 & 0 & x_4(k|k) & x_5(k|k) & 0 \\
0 & 0 & 0 & 0 & 0 & w_{44} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & w_{45} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & w_{54} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w_{\alpha}
\end{bmatrix} \tag{4.4-9}$$

The terms d_1 to d_3 are the input coefficients for the wind disturbance α_w. The state equations by which the conditional estimate is computed are

$$x(k+1|k) = A(k|k)\,x(k) + B\,m(k) + \Gamma\,\alpha_w(k|k)$$

$$y(k) = C\,x(k|k) + L\,v(k) \qquad (4.4\text{-}10)$$

where the matrices A, B, Γ, C are given by

$$A(k|k)=\begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 \\ 0 & 0 & 0 & a_{44}(k|k) & a_{45}(k|k) \\ 0 & 0 & 0 & a_{54}(k|k) & a_{55}(k|k) \end{bmatrix} \quad B=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} \quad \Gamma=\begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & y_1'(x_\phi) & 0 \\ 0 & 1 & 0 & 0 & y_1'(\dot{x}_\phi) \end{bmatrix}$$

and the disturbance α_w is given by

$$\alpha_w = \alpha_w(k|k) + \sigma_\alpha\, u(k) \qquad (4.4\text{-}11)$$

where u(k) is a unit variance random variable and α_w is the estimate of the average wind as plotted in Figure 4-9 for the nominal case.

Early runs with the adaptive system indicated that the convergence is highly dependent upon the initial values of the augmented covariance matrix. Choosing the initial value to be the identity matrix, a procedure which worked well in the other systems, invariably caused machine overflow. Apparently such a choice assigns too large an error to the parameter estimates, causing the δa terms to grow rapidly. Likewise, after this problem was solved, it was found that the estimator performance seemed much more sensitive to the values of the input noise gains for the parameters (γ_p) than the weighting term (w). Exactly why this should be so is not readily apparent. In any event Figure 4-17 shows the performance of the adaptive system for a -20% bending frequency. Due to program storage limitations the feedforward term was dropped in this system. It could have been added by using tape storage, however the resultant program slowdown was felt to be too much. Further, the effect of the feedforward was known to cause a decrease in the steady offset, not the vibration which was of primary interest from the identification point of view.

The reduction in bending vibration is striking. Notice that the system requires about 10 seconds to identify

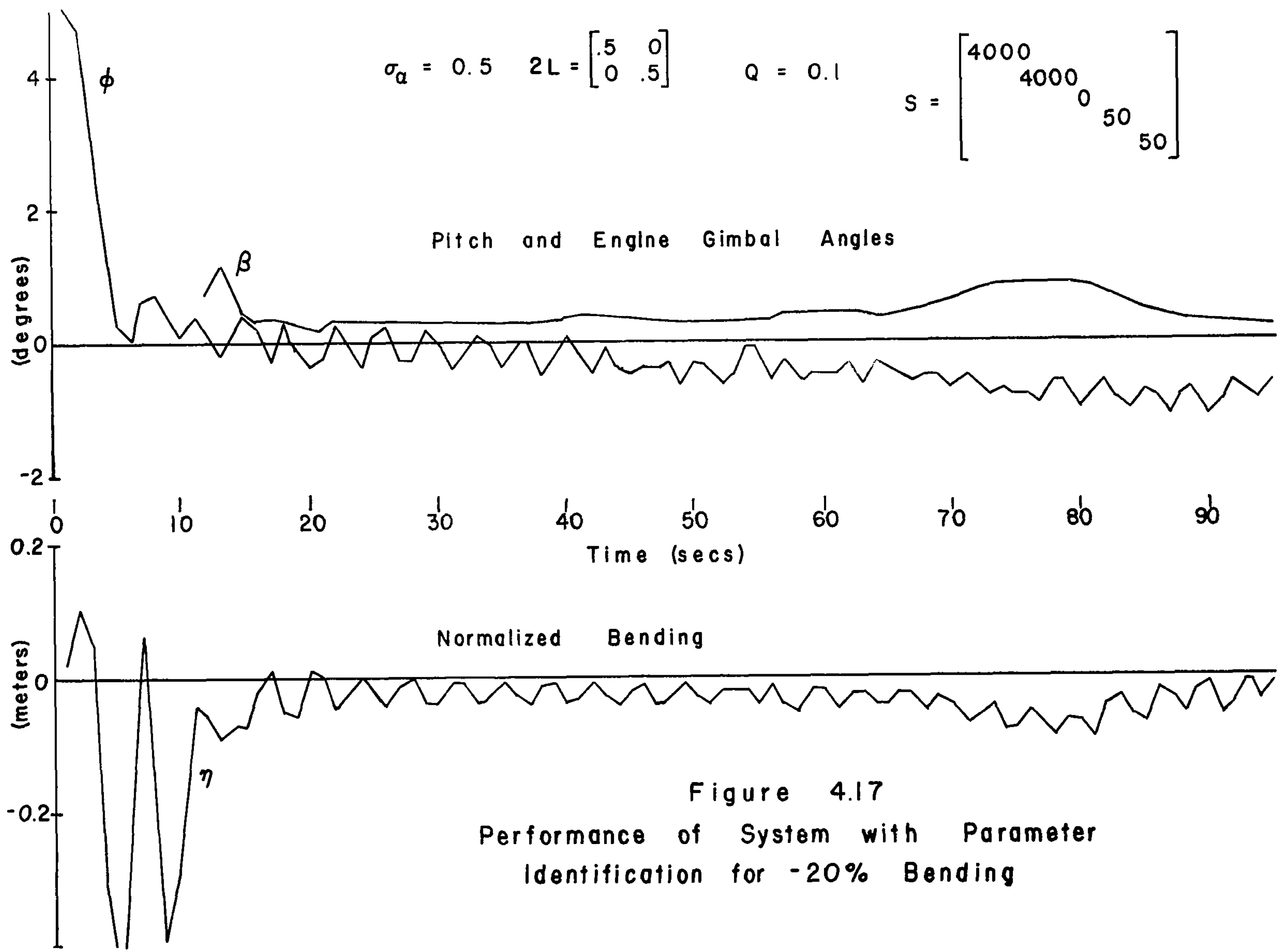

Figure 4.17
Performance of System with Parameter Identification for -20% Bending

the off nominal frequency after which the vibrations are well damped. However, the bending is not quite so good as in the nominal case and there is some additional pitch error. Interestingly, at the end of the flight there is a small vibration in the bending mode in either case. This is probably due to the predominant sensor noise when the pitch angle falls off to zero.

4.5 Discussion of Computer Results

The difference in performance of the system with and without adaption demonstrates the effectiveness of the synthesis procedure which has been developed. With the experience gained from the many computational runs which were made it is now possible to make some remarks about the results obtained.

1. The difficulty of the control portion of this problem has probably tended to obscure the identifier characteristics. Inspection of the run log in Appendix D shows that considerable time was spent choosing weighting factors. This is a result of the plant description. Less complex plants, without parallel branches, for instance, would present less of a challenge in the choice of weighting factors.

2. Like conventional optimal control, the weighting factors do not adequately express the subjective criteria of goodness which the designer has in mind. In particular, from a subjective viewpoint, the control and estimation problem cannot be separated. One of the biggest difficulties seems to be the propagation of noise through the feedback when the gains are increased in order to improve the performance.

3. Amplifying on item 2, it is apparent that the designer will seek a system which is insensitive to variations in the assumed noise levels. The limited experience gained from this example indicates that such a goal will be in conflict with achieving high control system accuracy. Therefore the weighting matrices cannot be picked by observation of a noiseless system.

4. Not only did the feedforward term reduce the offset error but it also reduced the bending vibration. It is now felt that the gains should have been chosen with the feedforward term included, rather than the method used in this study.

5. The sensitivity of the parameter identifier to the initial values of the covariance matrix was unexpected and resulted in considerable delay. This is in

contrast to the state estimate whose initial covariance seemed to have little effect upon performance after the first few time points.

6. The insensitivity of the parameter identifier to the weight matrix was also unexpected. Unfortunately the time consumed by the difficult problem of choosing control weighting factors and the fact that this was the most complex of the three simulations left little time for an exhaustive study of the influence of the fictitious noise gains and weighting factors upon the estimator. For an investigation of this sort it would seem advisable to postulate a simple model, preferably time invariant. In this way the effects of parameter identification can be easily studied apart from the control problem.

7. From the engineering viewpoint, the complexity of the adaptive system studied here leaves something to be desired. An alternate approach to problems of this type might be to postulate a controller composed of a Least Squares estimator and a gain matrix. Parameter optimization or noise sensitivity analysis could then be carried out on this structure to obtain a de-sensitized control system. Although this appears to be a step back in the sense that parameter adjustment is a well known

technique, it has the appeal of simple realization. (This is the approach advocated by Horowitz[14] in a way.)

8. Due to the computational complexity, the one step approximation of the new control law has not been evaluated against a complete recomputation over the interval. At this time it can only be concluded that the proposed adaptive control law worked in this case.

SECTION V

CONCLUSION

5.1 Conclusions

A method of synthesizing an adaptive controller for linear, discrete, time-varying systems has been developed. The development is based upon the assumption that the system will be divided into a feedback gain matrix operating upon state estimates supplied from a Least Squares type filter and a parameter identifier. The parameter identifier is to make new parameter estimates based upon observation of the normal input and output of the system. The plant is subject to state disturbances and the available measurements are noisy.

Incorporation of deterministic disturbances, or stochastic inputs with non-zero means, is easily handled by a variation on the conventional Dynamic Programming approach. The necessity of a Least Squares estimator is also seen to follow directly from the use of a quadratic performance index. The resulting system differs from the usual regulator only in the appearance of a feedforward term. The filter equations may be solved as a set of recursive relations. For non-adaptive systems the filter constants can be obtained from an off line computation since they are not a function of the measurements.

On line parameter identification is shown to require a time varying controller and excitation of all modes of the system. For both state and measurement noise Least Squares techniques cannot be applied directly. Adjoining the unknown parameters as additional states results in a non-linear estimation problem. Recursive relations, of the Least Squares type, can be obtained for such systems by considering a linearized model for the error propagation through the plant. For a linear plant the augmented equations are bi-linear and the error propagation is accurately described whenever the relative errors in the estimation are small with respect to the parameters themselves.

It is proposed that the new parameter estimates be used to alter the control law by making a one step calculation from the stored nominal Ricatti matrix. The example shows that the proposed method is workable. No meaningful conclusions about the general efficacy of the proposed adaption can be drawn until a comparison to the continuously recomputed optimal control is made. Such a comparison would be most easily made for fixed parameter plants.

The computational requirements of the augmented state approach are not trivial. For the five states and five parameters estimated in the example the augmented state vector has ten elements, or twice the dimension of the original state. Since most of the calculations involve square matrices the computations go up by a factor of four. Further, the augmented covariance propagation cannot be computed off line because the transition matrix is a function of the previous estimates and therefore of the actual previous measurements.

An attractive alternate, from the computational standpoint, would be to postulate a structure comprised of a Least Squares type filter and a gain matrix. Sensitivity analysis could then be applied to obtain a closed loop parameter insensitive system. This would have the advantage of requiring no on line calculations.

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SECTION VI

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Appendix A

Programming Techniques

A brief discussion of some of the less obvious numerical methods employed in the simulation of the control system and booster are given here.

A-1 Noise Generator

The white Gaussian random noise was simulated using a modified multiply sequence method. An equidistributed sequence of numbers X_n lying in the range (0, 1) are first generated using the recursive relationship

$$X_{n+1} = \{NX_n + \theta\} \qquad \text{(A1)}$$

where { } denotes "fractional part of". Franklin[12] has exhaustively analyzed such sequences and obtained many important results, the following theorem being among them.

Theorem: "For almost all X_0 , the sequence (given by A1) has an auto-correlation function $R(\tau)$ such that $R(\tau) \rightarrow 0$ uniformly in τ for $\tau \neq 0$ ".

The proof is quite involved and is given in Franklin's paper. From a computational standpoint the statistical accuracy is enhanced by causing N to be as large as possible for the given machine word length. The number θ is arbitrary and is used to prevent a continuing sequence of zeros from occurring.

From equation A1, the actual sequence generated in a simulation is entirely determined by X_0 the initial value or priming used in the generator. This allows the same random sequence to be repeated if desired for different runs. The advantage of the computational sequence of equation A1 over stored random digits tables from the storage viewpoint is obvious. But, the same paper also shows that even choosing θ transcendental is not sufficient to guarantee an equidistributed sequence. However, for computational purposes it is the most convenient.

Box and Muller[4] have found an ingenious method of converting the equidistributed distribution into a normally distributed one via a transformation. A straightforward transformation must use the error function, which itself is difficult to compute. However, their result is as follows:

Theorem: "Let X_1 and X_2 be independent random variables from the same rectangular density function on the interval (0, 1). Then the pair u_1, u_2 related to X_1 and X_2 by

$$u_1 = \sqrt{-2 \ln X_1} \cos 2\pi X_2 \qquad \text{(A2)}$$

$$u_2 = \sqrt{-2 \ln X_1} \sin 2\pi X_2 \qquad \text{(A3)}$$

are a pair of independent random variables from the same normal distribution with zero mean and unit variance."

As before, the proof is rather involved, however the authors also give an interesting heuristic argument to illustrate how they arrived at this transformation. If u_1 and u_2 are thought of as distances on orthogonal axes and assumed drawn from the same normal distribution then the variable

$$\theta = \tan^{-1} \frac{u_2}{u_1} \qquad \text{(A4)}$$

has its density function uniformly distributed over the interval 0 to 2π. The inverse of the transformation (A2) and (A3) is

$$X_2 = \frac{1}{2\pi} \tan^{-1} \frac{u_2}{u_1} \qquad \text{(A5)}$$

$$X_1 = e^{-\frac{1}{2}(u_1{}^2 + u_2{}^2)} \qquad \text{(A6)}$$

Therefore if u_1 and u_2 are normally distributed X_2 is uniformly distributed. The square of the radius $r^2 = u_1{}^2 + u_2{}^2$ has a Chi square distribution, therefore X_1 also has a uniform distribution.

More precisely, for a continuous transformation

$$y = f(X) \tag{A7}$$

the density function of y is related to that of X by

$$p_2(y) = p_1\left(X = f(y)\right) \; |J| \tag{A8a}$$

Since the joint density function of X_1, X_2 is

$$p(X_1, X_2) = 1 \quad X_1, X_2 \in (0, 1) \tag{A8b}$$

and J is the Jacobian of equations (A5) and (A6) the joint density of u_1, u_2 is

$$p(u_1, u_2) = \frac{1}{2\pi} e^{-\frac{1}{2}(u_1^2 + u_2^2)}$$

$$= \left[\frac{1}{\sqrt{2\pi}} e^{-\frac{u_1^2}{2}}\right]\left[\frac{1}{\sqrt{2\pi}} e^{-\frac{u_2^2}{2}}\right] \tag{A9}$$

The variables u_1, u_2 are therefore normally distributed.

A-2 Plot Routine

Because the computer was used as a simulation device rather than a data processor, numerical output data is of less interest than curves of the familiar analog type. These were generated through a special PLOT subroutine.

Briefly, the plot is generated by quantizing the variable to be plotted into approximately 100 levels. A symbol is then printed in the corresponding print position as a point on the curve. The subroutine allowed simultaneous plotting of up to eight curves, together with time axis and scale factor information.

Appendix B

The Continuous Case

The continuous case system equations may be obtained by analogy to the discrete case. No attempt at rigor will be made here, rather the corresponding results are simply indicated.

The controller has been shown[15] to be comprised of the optimal feedback gains and the continuous Kalman filter[17]. For deterministic disturbances, the plant equations are

$$\dot{X} = AX + Bm + d(t) \tag{B1}$$

The corresponding performance index to be minimized is

$$J = \int_{t_0}^{T} \Big[\frac{1}{2} X^T(\sigma)\ S(\sigma)\ X(\sigma) + \frac{1}{2} m^T(\sigma)\ Q(\sigma)\ m(\sigma)\Big]\, d\sigma \tag{B2}$$

The resultant control equation is obtained using Calculus of Variations and adjoining the plant equation (B1) to the performance index. The resulting equations are

$$m(t) = -Q^{-1} B^T \lambda(t) \tag{B3}$$

$$\dot{\lambda}(t) = -A^T \lambda(t) - SX(t) \tag{B4}$$

$$\lambda(T) = 0 \tag{B5}$$

If a terminal weighting matrix R is included then the boundary condition on the multiplier is

$$\lambda(T) = RX(T) \tag{B6}$$

which is equivalent to an impulse weighting of S at time T. To obtain a feedback and feedforward control assume a solution for the multipliers of the form

$$\lambda(t) = P(t)\, X(t) + \xi(t) \tag{B7}$$

Substitution of (B7) into (B4) and grouping of terms gives the resulting separated equations

$$-\dot{P}(t) = A^T P + PA + S - PBQ^{-1} B^T P \tag{B8}$$

$$-\dot{\xi}(t) = P\, d(t) - PBQ^{-1} B^T \xi(t) + A^T \xi(t) \tag{B9}$$

with boundary conditions

$$\xi(T) = 0, \quad P(T) = R \tag{B10}$$

The estimator for states and parameters is the continuous analog of the discrete filter. The noisy system is described by

$$\dot{X}(t) = A\, X(t) + Bm(t) + \Gamma\, u(t) + d_1(t) \tag{B11}$$

$$z(t) = C\ X(t) + L\ v(t) + d_2(t) \tag{B12}$$

For simultaneous estimation of states and parameters the continuous version of the Wiener-Kalman filter is used based upon the linearized plant model. Let X_A denote the augmented state vector and Γ_p the effective noise driving the parameters. In the limit as Δ t goes to zero the continuous equivalent of the parameter transition equations is

$$\frac{da_i}{dt} = k_i \left[\bar{a}_i - a_i\right] + \gamma_p\ u_i \tag{B13}$$

The linearized version of (B11 and (B12) is

$$\delta\dot{X}_A = \begin{bmatrix} \delta\dot{X} \\ \delta\dot{a} \end{bmatrix} = \left[\begin{array}{c|c} \hat{A} & \hat{X} \\ \hline 0 & -k_i \end{array}\right] \begin{bmatrix} \delta X \\ \delta a \end{bmatrix} + \left[\begin{array}{c} 0 \\ \hline k\bar{a} \end{array}\right]$$

$$+ \left[\begin{array}{c|c} \Gamma & 0 \\ \hline 0 & \Gamma_p \end{array}\right] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{B14}$$

or in terms of augmented matrices

$$\delta\dot{X}_A = A_A\ \delta X + B_A\ m_A + \Gamma_A\ u_A \tag{B15}$$

So that the estimator equations are

$$\frac{d\hat{X}_A}{dt} = \hat{A}\ \hat{X}(t) + \psi(t) \left[z(t) - C\ \hat{X}(t)\right] \tag{B16}$$

$$\psi(t) = P(t)\, C_A^T\, (LL^T)^{-1} \tag{B17}$$

$$\frac{dP}{dt} = A_A\, P(t) + P(t)\, A_A^T - PC_A^T\, LL^T\, C_A\, P + \Gamma_A \Gamma_A^T \tag{B18}$$

where

$$C_A = \left[C \,\middle|\, 0 \right] \tag{B19}$$

In the case of gain modifications as a result of new parameter estimates, the analogy is not so clear. For the discrete case one step ahead was a finite time interval and therefore a first order gain perturbation effect could be obtained. In the continuous case the Riccati matrix is at the same time instant. Therefore unless some method of extrapolating ahead a short interval and recomputing the gains backward is used it is not clear how the optimal gains ought to be altered as a function of the new parameter estimates. Indeed this is an important general question pertaining to adaptive control which must be investigated further.

Appendix C

Program Details for the Launch Vehicle Problem

As described in Section 4.2 the continuous description of the vehicle was converted to a discrete one via a digital computer program. The output of this conversion program was a set of cards, punched in an A format, containing the non-zero entries for the matrices A, B, C, Γ and the angle of attack due to the wind. This deck constituted the vehicle description. Similar decks were also made up for ± 20% lift coefficient deviations to provide off nominal vehicle representations.

Three types of programs were run. Each used basically the same subprograms but the main program was varied and some of the subprograms were also modified. The three programs were

1. Feedback control only. No parameter identification.
2. Feedforward plus feedback. No parameter identification.
3. Feedback control only. Parameter identification and control adaption.

Each of the three used the same type of subprograms although they differed somewhat from version to version. The subprograms used and their general purpose were

1. Function WNØISE - Generates an independent normal random variable of zero mean and unit variance. This is the noise generator referred to in Appendix A.

2. Subroutine UNPACK - To save cards the vehicle description was condensed to contain only the non-zero elements and placed in a large array, usually called "STORE", having time as one of the indexes. UNPACK recovered the A, B, C, Γ matrices and α_w from the array at a specified time point.

3. Subroutine STATE - Solved the state equations

$$\begin{aligned} x(k+1) &= Ax(k) + Bm(k) + Du(k) \\ y(k+1) &= Cx(k+1) \end{aligned} \tag{C1}$$

for the next x and y.

4. Subroutine GAIN - The generic name for the program computing the feedback gains (and the feedforward). GAIN computed one time step each time it was called, computing the new gains and new Riccati matrix.

5. Subroutine ESTIMD - Solves the Kalman filter equations for the correction matrix $\psi(k + 1)$ and the new covariance matrix from the old covariance. For the adaptive system this routine was called PSI since it dealt with augmented matrices.

6. Subroutine PLOT - Rather than print out the data numerically, PLOT produced the equivalent of an analog strip chart recording of the variables of interest in the problem.

Actually, programs 1 and 2 differed only in the computations in subroutine GAIN and in the control computation. Programs 1 and 2 each had two versions. The first version used the same vehicle data for computing the gains and actually flying the vehicle. The second version computed the gains from the first data set but read in a second set for use as the actual vehicle coefficients. The estimator in the second version used the data from which the gains were computed. Because the two programs differ only in the gain computation separate flow charts of them will not be presented. However, the sequence of computations for the two versions of each program is sufficiently different to warrant separate charts. For this reason flow charts are presented of version one, colloquially referred to as "BCP" and version two, called "BCP + 2" are given in Figures C1 and C2.

The "BCP" program proceeded in a straightforward manner. Control cards specifying the print out frequency and so on were first read followed by the vehicle data pack and the matrices Q and S. The gains are computed in a loop using the UNPACK routine to get the data and the GAIN routine for the computations. The resulting gains are stored back in the large array STORE. When the gains have been computed for the entire interval the data cards specifying initial conditions and noise amplitudes are read. Initial conditions and constants used are then printed and the flight is computed inside a second large loop. Within the flight loop the data is again unpacked for the current time point and the next control computed from the state estimate and the unpacked gain matrix. Prior to actually taking the step a status printout may be made and the data for the plotter is stored. The step is taken and a new measurement computed. This new measurement is processed by the estimator to yield a new state estimate and the loop is repeated until the terminal time. After the last point is computed the plotter prints out the flight data. The entire program is inside a program

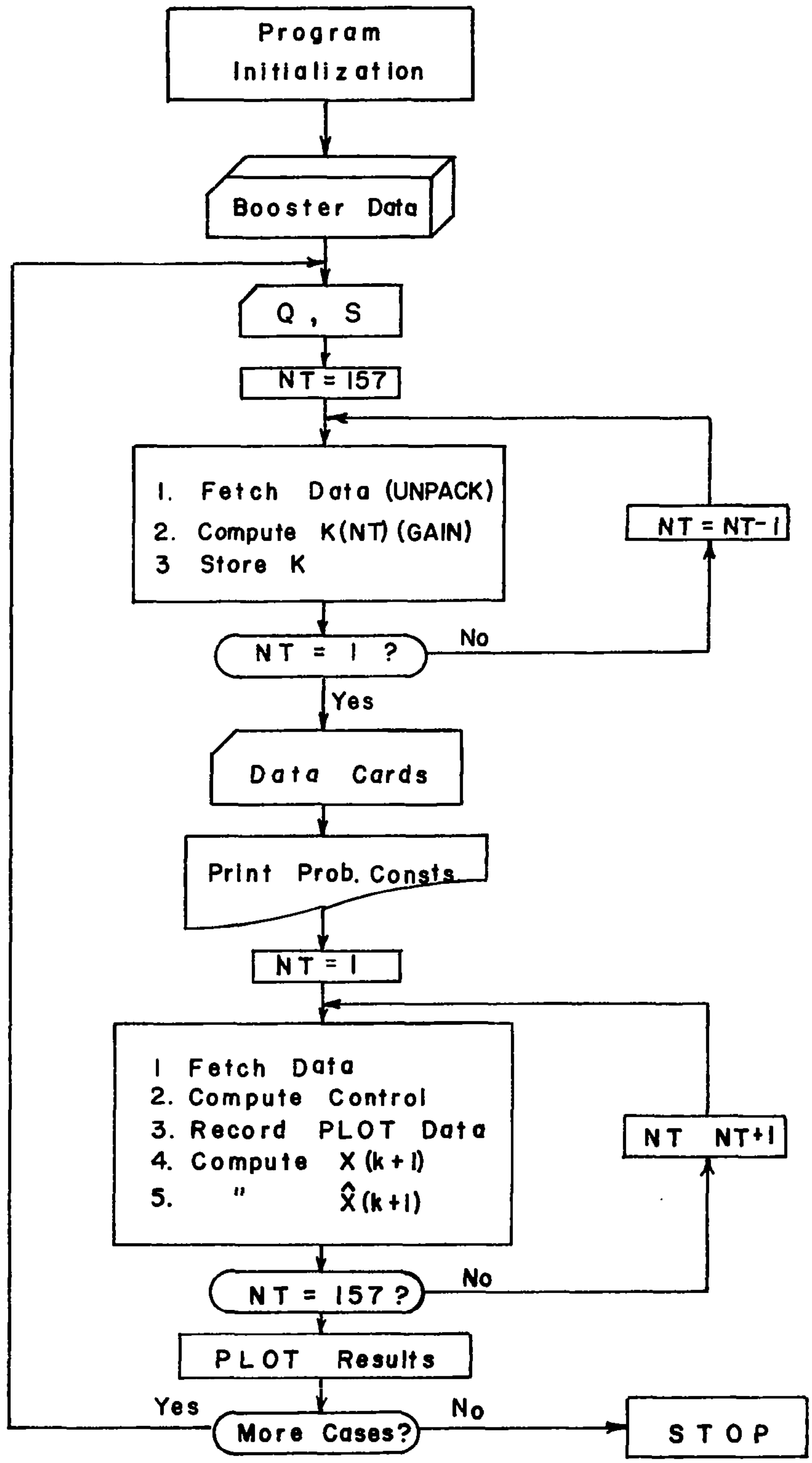

Figure C1 BCP Program Flow Chart

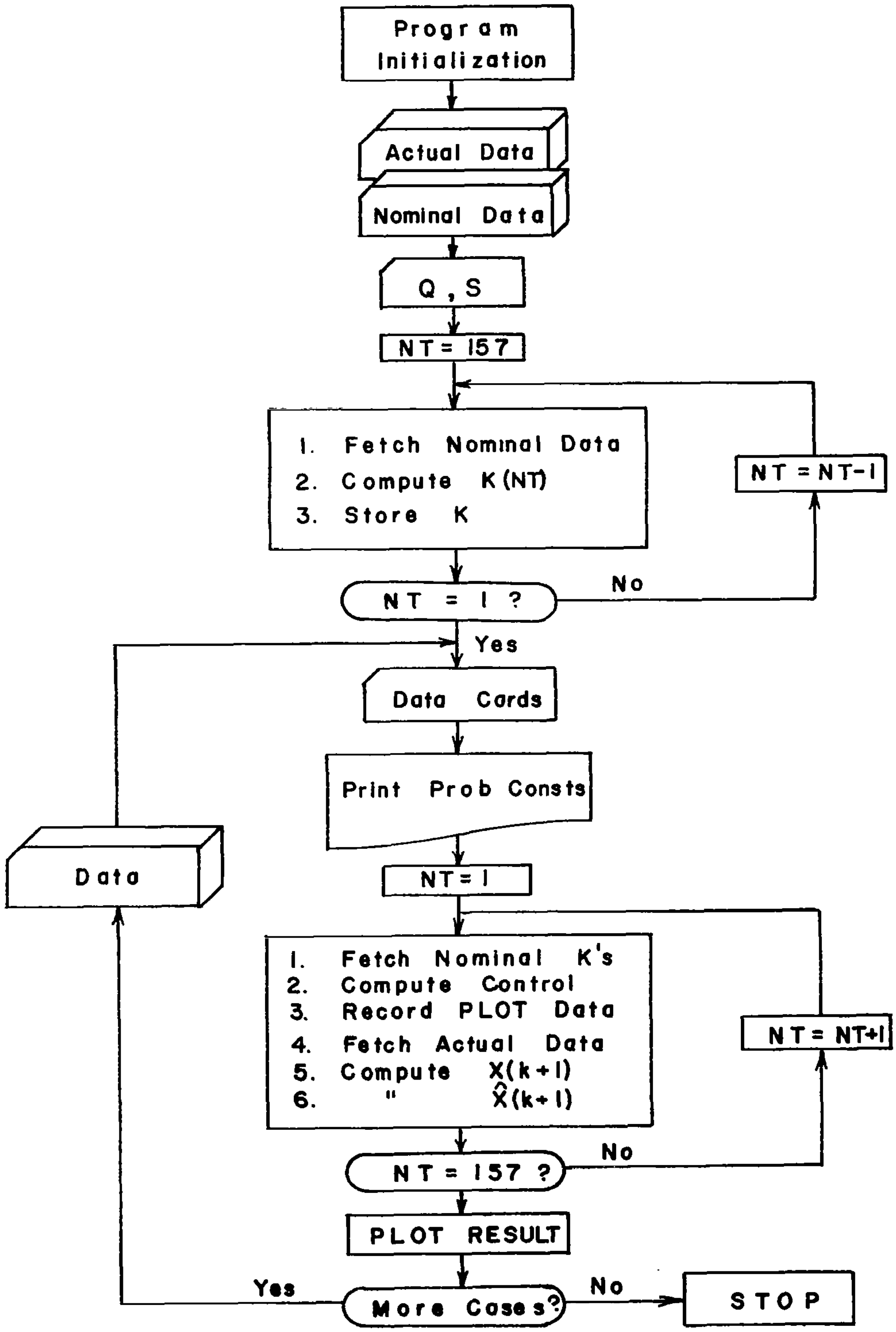

Figure C2 BCP + 2 Flow Chart

loop and may be re-run with new data.

The "BCP + 2" program was written to evaluate system performance when the actual vehicle parameters differed from those used in the gain computation and the estimator. In this case two vehicle data sets are read in. The gains are computed from the first data set as are the controls and the estimator outputs. However, the actual state is computed using the second data set which can represent off nominal vehicle performance. Notice that successive cases use the same data set No. 1 but read a new second data set. Also the gains are not recomputed for succeeding cases so that the program can only evaluate the effect of various types of abnormal conditions for a given controller.

Finally, the adaptive program flow chart is shown as Figure C3. As before the control cards and first data set are read in and the gains computed. However it is not the gains that are stored but the Riccati matrix which occupies more room. The Riccati matrix is temporarily put into the second data set array which is blank. When the entire interval is computed the Riccati matrix, together with all vehicle data is recorded on tape in the forward time direction. This was necessitated by the extra space required for the Riccati matrix storage. The tape is then rewound and the second data set read in to be used for the actual vehicle data. The parameter estimates are primed with the first values on the tape, the program data cards are read in and the flight loop commences. The sequence of computations is the same but now the estimator is using the augmented equations. Multiple cases may be run for the same set of nominal parameters.

Appendix D

Digital Simulation Record

The following is a chronological history of the runs made on the digital computer to evaluate the vehicle control system. Not included are the early program development runs which were used for checkout purposes. The runs were divided up into blocks which are composed of related runs. For each run the values of Q and S are given or indicated and the noise powers are given by stating the variance of the wind induced angle of attack and the diagonal values of the L matrix. If the actual noise level differed from the assumed level then the actual level is given with the word actual. Finally, a descriptive phrase or sentence indicates the performance of the system. In order to save time few intermediate results were printed. Output consisted of the digital plot generated by the machine. The evaluations were made from this plot.

All runs were made with an initial pitch error of 5^{o} and a bending deflection of .02 meters. All other states were zero. The covariance matrix, P, was initialized as the unit matrix. For the adaptive system, the augmented P matrix was initialized as

$$P = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right] \qquad \text{(D1)}$$

Also, for the adaptive runs, a choice of parameter noise, γ_p , for each of the five parameters and a value of the weighting factor w had to be made. These choices are also indicated.

The descriptive phrases, for brevity, are invariably subjective evaluations. For a fuller appreciation of any given run it would be necessary to inspect the actual output plot. Unless otherwise noted only the diagonal terms of matrices are given, the others being zero.

Block I - These runs were all made using nominal data. The controller has no feedforward.

1. Q = 0.1 S = 2000, 1000, 0, 5, 2.5

   sigma wind = 1.0 L = .1, .1

   Result: High vibration in bending mode

2. Q = 0.1 S = 2000, 1000, 0, 5, 2.5

   sigma wind = 1.0 L = .5, .5

   Result: High vibration, but better than I-1.

3. Q = 0.1 S = 2000, 1000, 0, 50, 25

   sigma wind = 1.0 L = .5, .5

   Result: Better than I-2 with respect to bending but I-2 has tighter pitch control and larger β.

4. Q = 0.1 S = 2000, 1000, 0, 5, 2.5

   sigma wind = 1.0 L = .5, .5

   actual L = 0.1, 0.1

   Result: Severe vibration, like I-1

5. Q = 0.1 S = 2000, 1000, 0, 50, 25

   sigma wind = 1.0 L = .5, .5

   actual L = 0.1, 0.1

   Result: About the same as I-3 in pitch. Very little vibration.

6. Q = .1 S = 2000, 1000, 0, 50, 25

   sigma wind = 0.2 L = .1, .1

   Result: Roughly equivalent to I-5

7. Q = .01 S = 2000, 1000, 0, 50, 25

   sigma wind = 0.2 L = .1, .1

   Result: Identical to I-6 up to 2 decimal places in all quantities.

Block II - These runs were made to try and evaluate the effects of off diagonal terms in S, and weighting of angle of attack. The only non-zero off diagonal term was S_{14} a weighting of $\phi \times \eta$ and, because S is symmetric, S_{41}.

1. Q = 0.1   S = 2000, 1000, 5, 50, 25

sigma wind = .5   L = .25, .25

Result: High vibration, very poor.

2. Q = 0.1   S = 2000, 1000, 100, 50, 25

sigma wind = .5   L = .25, .25

Result: Unstable for first 90 sec.

3. Q = 0.1   S = 2000, 1000, 0, 50, 25, S_{14} = 5

sigma wind = .5   L = .25, .25

Result: Good performance. Pitch error at 80 secs is .7°.

4. Q = 0.1   S = 2000, 1000, 0, 50, 25, S_{14} = 500

sigma wind = .5   L = .25, .25

Result: Unstable nearly everywhere.

5. Q = 0.1   S = 2000, 1000, 0, 50, 25

sigma wind = .5   L = .25, .25

actual wind = .01   actual L = .05, .05

Results: Good performance, similar to II-3 but smoother due to low noise.

6. Q = 0.1   S = 2000, 1000, 0, 50, 25

sigma wind = .5   L = .25, .25

actual wind = 0   actual L = 0, 0

Results: Like II-5. Maximum pitch error is $.7^{o}$ due to steady wind shear.

Block III - In an effort to further tighten the pitch loop the corresponding S matrix terms were raised.

1. Q = 0.1 S = 10000, 5000, 0, 50, 25 S_{14} = 5

   sigma wind = .5 L = .25, .25

   Results: Improved performance over II-5.

   Pitch error at 80 secs = 0.4^{o}

2. Q = 0.1 S = 10000, 5000, 0, 50, 25 S_{14} = 5

   sigma wind = .5 L = .25, .25

   actual wind = .1 actual L = .1, .1

   Results: Performance nearly the same as III-1.

3. Q = 0.1 S = 20000, 10000, 0, 50, 25 S_{14} = 50

   sigma wind = .5 L = .25, .25

   Results: No pitch improvement but bending worsens.

4. Q = 0.1 S = 20000, 10000, 0, 50, 25 S_{14} = 150

   sigma wind = .5 L = .25, .25

   Results: High vibration.

5. Q = 0.1 S = 20000, 10000, 0, 50, 25 S_{14} = 50

   sigma wind = .5 L = .25, .25

   actual wind = .1 actual L = .1, .1

Results: Very poor pitch and high vibration.

Block IV - In these runs the gains were computed with nominal data but the actual flight used off nominal vehicle data as indicated.

1. $Q = .1$ $S = 10000, 5000, 0, 50, 25$ $S_{14} = 5$
   sigma wind = .5 $L = .25, .25$
   Actual vehicle had +20% higher bending frequency.
   Results: High vibration.

2. Same as IV-1 but for -20% bending
   Results: High vibration.

3. Same as IV-1 but actual vehicle had +20% lift coefficient and nominal bending.
   Results: Good performance. At 80 sec pitch error = 0.4°.

4. Same as IV-1 but actual vehicle had -20% lift coefficient and nominal bending.
   Results: Similar to IV-3.

5. $Q = 0.1$ $S = 2000, 1000, 0, 50, 25$ $S_{14} = 5$
   sigma wind = .5 $L = .25, .25$
   Actual vehicle had +20% bending frequency.
   Results: High vibration.

6. Same as IV-5 but with -20% bending

Results: High vibration.

7. Same as IV-5 but with nominal bending and +20% lift coefficient.

Results: Good performance, control effort appears to be less.

8. Same as IV-5 but with nominal bending and -20% lift coefficient.

Results: Similar to IV-7, but control effort is greater.

Block V - This set was run using Q = 0.1 and S = 10000, 5000, 0, 50, 25 S_{14} = 5 for all cases. The objective was to evaluate the control for off nominal noise levels and vehicle parameters. The design noises in each case were sigma wind = .5, L = .25, .25. The actual noises were the same unless otherwise noted.

1. Actual vehicle had +20% bending

Results: High vibration.

2. Actual vehicle had -20% bending

Results: High vibration.

3. Actual vehicle had +20% lift

Results: Good. Compares to II-3.

4. Actual vehicle had -20% lift

   Results: Good. Similar to V-3.

5. Nominal vehicle. Actual L = .05, .05

   Results: Good performance. Better than II-3.

6. Nominal vehicle. Actual wind sigma = .1

   Results: Good. Better than II-3.

7. Data card error invalidated this run.

8. Nominal vehicle. Actual wind sigma = 1,

   actual L = 1.0, 1.0

   Results: Some increase in bending over II-3 and slight vibration.

Block VI - These runs evaluated other choices of the weighting matrices Q and S.

1. Q = 0.1 S = 5000, 10000, 0, 50, 100 $S_{14} = 10$

   sigma wind = .5 L = 0.1, 0.1

   Results: Good performance, but not radically better than others, lower slew rate.

   Pitch error at 80 sec is 0.5°.

2. Q = 0.1 S = 2000, 2000, 0, 50, 50 $S_{14} = 10$

   sigma wind = 0.5 L = 0.1, 0.1

   Results: Good performance about equal to II-3.

3. Q = 0.1 S = 2000, 4000, 0, 50, 50

sigma wind = 0.5 L = 0.1, 0.1

Results: Same as VI-2.

Block VII - These runs were made as a revalidation following a rewrite in the estimator subprogram which compressed the running time.

Block VIII - All of these runs were made with Q = 0.1 and S = 4000, 4000, 0, 50, 50. This is the set of weighting factors finally chosen throughout the remainder of the study.

1. sigma wind = .5 L = .25, .25

   Results: Good. At 80 secs pitch error is $.5^{\circ}$.

2. sigma wind = .5 L = .25, .25

   actual L = .05, .05

   Results: Performance not appreciably different from VIII-1.

3. sigma wind = .5 L = .25, .25

   actual wind = .1

   Results: Same performance as VIII-1.

4. sigma wind = .5 L = .25, .25

   Actual vehicle had -20% bending frequency.

   Results: Pitch control looks reasonable but there is a large vibration in the bending which does not damp out until after max q.

Block IX - All of these runs are for the adaptive system (i.e. containing the parameter identification). The first four resulted in program interrupts due to an exponential overflow in the covariance matrix. All had in common an initial unit covariance matrix P rather than the P given in D1.

5. Q = 0.1   S = 4000, 4000, 0, 50, 50

   sigma wind = .5   L = .25, .25

   WT = 0, 0, 0, 0, 0

   param gamma = 0, 0, 0, 0, 0

   P = 1, 1, 1, 1, 1, 0, 0, 0, 0, 0

   Results: This run duplicates VIII-1, as it should and therefore provides a validity check.

6. This run invalidated by a data card error causing

   P = 1, 1, 1, 1, 0, 0, 0, 0, 0, 0

All following runs used

Q = 0.1   S = 4000, 4000, 0, 50, 50

sigma wind = .5   L = .25, .25

as controller design data. Further, the initial covariance matrix was always that of equation D1 and all actual vehicle data had a 20% reduction in bending frequency.

7. WT = 0, 0, 0, 0, 0

   Param gamma = 0, 0, 0, 0, 0

   Results: This run duplicated VIII-4 as it should providing a further validity check.

8. WT = 0, 0, 0, 0, 0

   Param gamma = 0.1, 0.1, 0.1, 0.1, 0

   Results: Lower vibration than IX-7. Pitch is roughly the same.

9. WT = 0.1, 0.1, 0.1, 0.1, 0

   Param gamma = 0.1, 0.1, 0.1, 0.1, 0

   Results: About the same as IX-8.

10. WT = 0.2, 0.2, 0.2, 0.2, 0

    Param gamma = 0.1, 0.1, 0.1, 0.1, 0

    Results: Again little change from IX-8.

11. WT = 0.3, 0.3, 0.3, 0.3, 0

    Param gamma = 0.1, 0.1, 0.1, 0.1, 0

    Results: Some reduction in the bending over IX-10.

12. WT = 0.1, 0.1, 0.1, 0.1, 0.1

Param gamma = 0.1, 0.1, 0.1, 0.1, 0.1

Results: Slightly better than IX-11.

Block X - These runs were made with the feedforward term but without the parameter identification

1. Q = 0.1 S = 4000, 4000, 0, 50, 50

   sigma wind = .5 L = .25, .25

   Results: In comparison to VIII-1 the bending at max q is halved but there is little pitch reduction (maybe 10%).

2. Q = 0.1 S = 2000, 1000, 0, 50, 25

   sigma wind = .5 L = .25, .25

   Results: In comparison to block I runs the bending and pitch both are reduced.

3. Q = 0.1 S = 4000, 4000, 0, 50, 50

   sigma wind = .5 L = .25, .25

   actual wind = 0.1

   Results: Same performance as 1 but smoother pitch.

4. Q = 0.1 S = 4000, 4000, 0, 50, 50

   sigma wind = .5 L = .25, .25

   actual L = .05, .05

Results: Better estimator tracking, the vehicle performance is unchanged.

5. Q = 0.1 S = 4000, 4000, 0, 50, 50

   sigma wind = .5 L = .25, .25

   actual vehicle had -20% bending frequency

   Results: Pitch still performs well but there is high bending vibration.

6. Q = 0.1 S = 2000, 1000, 0, 50, 25

   sigma wind = .5 L = .25, .25

   Actual vehicle had -20% bending frequency

   Results: Like X-5 bending vibration increases but pitch is controlled.

Block XI - These are all evaluations of the adaptive system made with the actual vehicle bending reduced 20%. In all cases

Q = 0.1 S = 4000, 4000, 0, 50, 50

and the initial covariance as given in equation D1. The assumed noise levels were

sigma wind = 0.5 L = .25, .25

and the actual noises coincided unless noted otherwise.

1. WT = 0.5, 0.5, 0.5, 0.5, 0

   Param gamma = 0.1, 0.1, 0.1, 0.1, 0

Results: Tracking of estimator is quite good after first ten seconds. At maximum q pitch error is 1^{o}.

2. WT = 0.7, 0.7, 0.7, 0.7, 0

   Param gamma = 0.1, 0.1, 0.1, 0.1, 0

   Results: Same as XI-1.

3. WT = .3, .3, .3, .3, 0

   Param gamma = 0.2, 0.1, 0.2, 0.15, 0

   Results: The first 20 seconds have a somewhat higher bending than XI-2. Thereafter they are the same.

4. WT = 0.5, 0.5, 0.5, 0.5, 0

   Param gamma = 0.2, 0.1, 0.2, 0.15, 0

   Results: Very slight changes from XI-3.

5. WT = 0.7, 0.7, 0.7, 0.7, 0

   Param gamma = 0.2, 0.1, 0.2, 0.15, 0

   Results: Very slight changes from XI-4.

6. Actual wind = 0.1

   WT = 0.5, 0.5, 0.5, 0.5, 0

   Param gamma = 0.1, 0.1, 0.1, 0.1, 0

   Results: Virtually identical to XI-1.

7. Actual L = 0.5, 0.5

WT = 0.5, 0.5, 0.5, 0.5, 0

Param gamma = 0.1, 0.1, 0.1, 0.1, 0

Results: Performance is better than XI-1.

Smoother control action.

8. WT = 0.5, 0.5, 0.5, 0.5, 0.5

Param gamma = 0.2, 0.1, 0.2, 0.15, 1.0

Results: Performance is worse than XI-1 in bending estimation, resulting in increased vibration.

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